RISK MODELS IN THE CONTEXT OF EQUITY-LINKED LIFE INSURANCE

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Abstract

Equity-linked life insurance policies with an asset value guarantee are becoming more and more popular in Germany. In a series of papers, we have analysed the investment part of such policies in great detail. The present paper addresses the topic of calculating the risk premium (and hence the gross premium). It is remarkable that within our model, these aspects can be analysed independently.

Whenever the sum payable at death is random, the traditional actuarial methods can not be applied. Instead, a market evaluation approach — provided by the arbitrage pricing theory — is needed. We give empirical results for two popular products and three possible models for the sum payable at death. Here, the sensitivity of the gross premium with respect to different market scenarios and the age of the insured person is analysed. In particular, using a deterministic risk model, we investigate the difference between our market-driven evaluation principle and the traditional actuarial methods.

Keywords

Equity-linked life insurance, asset value guarantee, risk models, risk premium, arbitrage pricing theory.

1 Introduction

The first guaranteed equity-linked life insurance policy in Germany was sold in 1996 by Standard Life, a Scottish insurance company. Up to now, there are five German providers for such equity-linked products. All of the products being discussed at the moment have a common feature: the insurance company guarantees to pay back a predetermined amount at maturity. In addition, the policy is linked — via the so called rate of index participation — to a German equity index, here the DAX30, a performance index. There also exists one product, where the policy is linked to a basket of different European indices. This, in principle, does not alter our discussion.

In a series of papers (Nonnenmacher (1998a), Nonnenmacher (1998b), Nonnenmacher and Ruß (1997), Nonnenmacher and Ruß (1999), Nonnenmacher and Schittenhelm (1997), Ruß (1998), Ruß and Schittenhelm (1998)), we have analysed such policies in detail. All these papers, however, concentrate on the financial aspects of the policy (taking in particular the legal issues into account), which, within our model, can be treated independently of the topic of calculating the risk premium associated with the additional sum payable at death, in general random. Whenever this additional sum payable at death is random, the traditional actuarial methods can not be applied. Instead, a market evaluation approach — provided by the arbitrage pricing theory — is needed. This topic is addressed in the present paper.

In Section 2, we briefly explain the product design of equity-linked life insurance policies. In particular, we look at two popular products that have been introduced and analysed in Nonnenmacher and Schittenhelm (1997) and Nonnenmacher and Ruß (1999), respectively. In Section 3, we then introduce three possible models for the sum payable at death and after stating our model for the economy in Section 4, we calculate the fair rates of index participation for both products in Section 5, and, in Section 6, we give a very general formula for calculating the risk premium.

In Section 7, we derive empirical results for the risk premium (and hence the gross premium) using explicit pricing formulas whenever available as well as extensive Monte Carlo simulation techniques. The sensitivity of the gross premium with respect to different market scenarios and the age of the insured person is examined in detail for both products. Furthermore, we compare our results to those derived with traditional actuarial methods for a deterministic risk model. All of our analysis is done for two models for the costs of the policy: one where all costs are stated explicitly and one where acquisition costs are "hidden" in a reduced rate of index participation. The latter is interesting for marketing the products.

Section 8 concludes with a summary and an outlook for further research, and in the appendix, we explain our simulation aproach in detail.

2 Product Design

We consider equity-linked life insurance policies with an asset value guarantee, a term of $T = 12$ years and annual net premium payments NP at the beginning of the first $m = 5$ years. According to German tax legislation, life insurance policies are only privileged if $T \geq 12$ and $m \geq 5$. In addition, further conditions have to be fulfilled, cf. Section 3.

The m net premiums NP are assumed to be equal. Furthermore, we assume that all costs associated with the product such as acquisition costs, management fees and the risk premiums RP (depending on age and gender of the insured person) are equally distributed over the first $m - 1$ years (i.e. on the times of premium payment $t = 0, 1, \ldots, m - 1$). This results in m equal annual gross premiums GP.

The net premiums are swapped into a single payment, that is used to buy a security (from an investment firm) that pays off the sum A_T at the time of expiration ($t = 12$). This sum contains a guaranteed sum G, usually written as $G = \sum_{i=1}^{m} NP(1+i_g)^{T-i+1}$, meaning that a certain guaranteed rate of interest $i_g \geq 0$ is earned on the m net premiums. In addition, A_T depends on the performance of an equity index over the term of the policy.

The market value at time t of the (savings part of the) policy is defined to be

$$V_t = \max[0, A_t + SW_t],$$

where A_t denotes the value at time t of a security that pays off A_T at time T and SW_t denotes the value at time t of the above defined swap contract from the insurance company's perspective. Note, that SW_t is always ≤ 0, hence $A_t + SW_t < 0$ might occur, and $SW_t = 0$ for $t \geq m$.

Our definition of V_t results from the fact, that we do not allow for negative market values of the policy as that would imply negative surrender values. In what follows, we assume, that the market value of the investment also is V_t rather than $A_t + SW_t$. This is only relevant in case of death or cancellation. In Nonnenmacher (1998a) and Nonnenmacher and Ruß (1999), we show how suitable options that are paid from the first net premium have to be used to create this market value. For simplicity of notation, we here assume that these options have zero value, cf. also Section 5.

In this paper, we look at two specific products with the following two payoff functions:

$$A_T^1 = NP \sum_{i=1}^{m} \prod_{j=i}^{T} \left(1 + \max\left[i_g, \frac{S_j - S_{j-1}}{S_{j-1}} x_1\right]\right)$$

$$A_T^2 = \max\left[NP \sum_{i=1}^{m} \left(1 + \max\left[\frac{\frac{1}{T-i+1}\sum_{j=i}^{T} S_j - S_{i-1}}{S_{i-1}} x_2, 0\right]\right), G\right]$$

Here, S_j denotes the value of the DAX30 (German equity index) j years after the policy was sold and $x_k > 0$ is called the rate of index participation of product $k = 1, 2$. The rate of index participation is the percentage rate by which the insured person participates in index gains. See Section 5 for details on the calculation and the existence of the rate of index participation. Note that the minimum payment guaranteed at expiration is the same for both products: $G = \sum_{i=1}^{m} NP(1 + i_g)^{T-i+1}$.

In product 1, compound interest is earned on the net premiums. The rate of interest earned in year j is calculated as x_1 times the DAX30 return in year j but no less than i_g. This procedure of locking in gains during the term of the policy is called cliquet version. This product has been introduced in Nonnenmacher and Schittenhelm (1997).

In product 2, for each net premium, the arithmetic averaging return of the DAX30 from the time of premium payment until maturity is calculated and weighted with x_2. The sum of all these returns plus the net premiums is compared to G and the higher amount is paid off. This product has been analysed in detail in Nonnenmacher and Ruß (1999).

A contract similar to our product 1 is sold in Germany, policies similar to product 2 are sold heavily e.g. in Switzerland. These two payoff patterns provide a deep insight in the techniques needed to analyse guaranteed equity-linked policies since for one of them, closed form pricing formulas are available, whereas numerical methods such as Monte Carlo simulation are required to analyse A_t^2. In our setting, there is a great variety of further possible payoff functions. A systematic overview is given in Ruß and Schittenhelm (1998). To keep our notation as simple as possible, we assume throughout the paper, that the policy is sold immediately after the date of the last balance sheet. In Germany, insurance companies have to provide such statements on a yearly basis. We furthermore assume, that if the insured person dies in year j, i.e. in $(j-1, j]$, the sum payable at death is paid at $t = j$. In the analysis to follow, we therefore only consider integer values for t. Note that in all our models the sum payable at death depends on the value of the investment and therefore, if the insured person dies in $(j-1, j]$, the exact sum is not known until $t = j$. If this is not desired, non-integer values have to be considered in our analysis, cf., e.g., equation (8).

3 Risk Models

For an insurance policy in Germany to be privileged with respect to taxes it is neccessary, that the sum payable at death D_t always exceeds at least 60% of the sum of all gross premiums, i.e. $D_t \geq 0.6mGP$. Furthermore, to match the definition of an insurance policy, it is neccessary, that $D_t \geq V_t \; \forall t$. Otherwise, if the insured person dies at time t, the insurance company makes a profit, i.e. the capital under risk is negative. Although this is not explicitly forbidden under German legislation, the supervisory authority did not allow such policies before the deregulation took place in 1994. Now, after the deregulation, insurance companies can offer policies without prior consent of the supervisory authority. Nevertheless, as far as we know, no German insurance company offers a policy with negative capital under risk. However, risk models like our model 2 (described below) with $\lambda_1 = 1$ and $\lambda_2 = 0.6$ are used. This means that the capital under risk can be equal to 0 after some years.

If the insured person dies, the insurance company sells the investment V_t that has been made to generate the sum payable at expiration. When calculating the risk premium, we therefore only need to consider the additional sum payable at death

$$D_t^e = D_t - V_t,$$

which is called the effective sum payable at death in what follows.

Model 1:

One way to achieve the two conditions stated above is to let $D_t = V_t + 0.6mGP$. In this model, however, the effective sum payable at death $D_t^e = D_t - V_t = 0.6mGP$ is deterministic and constant. Therefore, the risk premium could be calculated traditionally as the risk premium of a term life insurance with constant sum payable at death. This traditional way, however, leads to different results than an approach with modern methods in financial mathematics, cf. Section 7.

We now introduce two further models for the sum payable at death. For those, the traditional way of calculating the risk premium cannot be applied, since the effective sum payable at death D_t^e depends on V_t and hence on the stochastic stock price process.

Model 2:

$$D_t = \max[\lambda_1 V_t, \lambda_2 mGP]$$

If $\lambda_1 \geq 1$ and $\lambda_2 \geq 60\%$, the conditions mentioned above are fulfilled. In this case, we get

$$D_t^e = \max[(\lambda_1 - 1)V_t, \lambda_2 mGP - V_t].$$

In our calculations in Section 7, we let $\lambda_1 = 1.05$ and $\lambda_2 = 0.6$.

Model 3:

$$D_t = \max[V_t, \lambda mGP] + c$$

If $\lambda \geq 60\%$ and $c \geq 0$, the conditions again are fulfilled. Here, the effective sum payable at death is

$$D_t^e = \max[0, \lambda mGP - V_t] + c.$$

In our calculations in Section 7, we let $\lambda = 0.6$ and $c = 1000DEM$.

In models 2 and 3, the sum payable at death will in general be smaller than in model 1. This leads to a lower risk premium and hence a lower gross premium. As most investors are interested in a high return on capital in case of surviving the term of the policy, these models are preferable from a marketing point of view. Most of the policies sold in Germany use a model similar to our model 2.

Figures 1 - 3 show for a given path of V_t, the paths of D_t and D_t^e in our three Models.

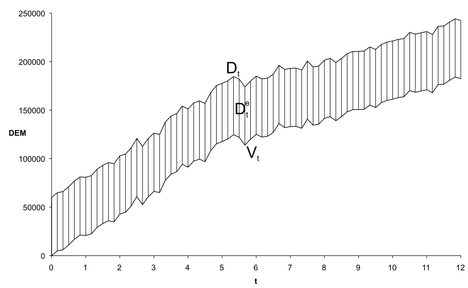

Figure 1: Risk model 1

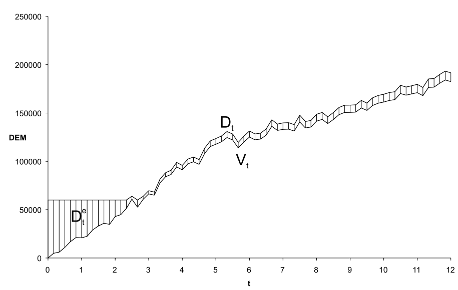

Figure 2: Risk model 2

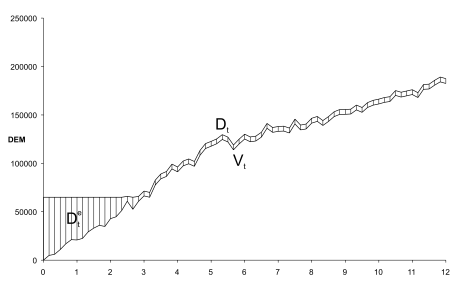

Figure 3: Risk model 3

4 The Model for the Economy

We will now state our model for the economy:

- The DAX30 follows a geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu(t)dt + \sigma dW_t,\ t \geq 0, \tag{1}$$

where W_t denotes a Wiener process on a probability space. Note that $\mu(t)$ is time dependent whereas σ is assumed to be constant (> 0).

The solution to the stochastic differential equation (1) is given by (cf. Karatzas and Shreve (1988)),

$$S_t = S_0 e^{\int_0^t \mu(s) - \frac{\sigma^2}{2} ds + \sigma W_t},\ S_0 > 0. \tag{2}$$

In particular, it follows that $\log \frac{S_{t_2}}{S_{t_1}} \sim N\left(\int_{t_1}^{t_2} \mu(t) - \frac{\sigma^2}{2} dt, \sigma^2(t_2 - t_1)\right)$ for $0 \leq t_1 < t_2$, where $N(m, v)$ denotes the normal distribution with mean m and variance v.

- The short rate process $r(t)$ is assumed to be deterministic and to fit the current, riskless term structure of interest rates, i.e.

$$\int_{t_1}^{t_2} r(t)dt = (t_2 - t_1) f_{t_1,t_2},$$

where f_{t_1,t_2} denotes the continuous, annualised forward rate for the period of time $0 \leq t_1 < t_2$.

According to Harrison and Pliska (1981), the value of A_t^k is given by

$$A_t^k = E_Q \left[e^{-\int_t^T r(s)ds} A_T^k \ \ |_t \right], \quad 0 \leq t \leq T,$$

where $E_Q[.\ |_t]$ denotes the conditional expected value under the information available at time t according to an equivalent martingale measure Q. The (unique) existence of such a measure Q is essentially equivalent to the assumption of a complete arbitrage-free market. In (1), $\mu(t)$ is substituted by $r(t)$ as a consequence of this transformation of measure and hence we have

$$\frac{dS_t}{S_t} = r(t)dt + \sigma dW_t, \tag{3}$$

or

$$S_t = S_0 e^{\int_0^t r(s) - \frac{\sigma^2}{2} ds + \sigma W_t}, \tag{4}$$

describing the evolution of the DAX30 in a risk-neutral world.

Within this model, explicit pricing formulas for $A_t^1,\ t = 0, \ldots, T-1$ can be derived: Let

$$c_t(\alpha, \beta, \gamma) = \alpha N(d_1) - \beta e^{-\int_t^{t+\gamma} r(s)ds} N(d_2)$$

denote the time t-value (according to the well-known Black-Scholes formula) at time t of a European call option on the DAX30, maturing at time $t+\gamma$, with a current index value $S_t = \alpha$ and strike price β. Here,

$$d_1 = \frac{\log \frac{\alpha}{\beta} + \int_t^{t+\gamma} r(s) + \frac{\sigma^2}{2} ds}{\sigma\sqrt{\gamma}}, \quad d_2 = d_1 - \sigma\sqrt{\gamma}, \text{ and } N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds.$$

Let furthermore

$$R_j = \frac{S_j - S_{j-1}}{S_{j-1}} x_1,$$

$$v_j = 1 + \max[i_g, R_j]$$

and

$$w_j = 1 + i_g + (x_1 + i_g) e^{f_{j-1,j}} c_{j-1}\left(\frac{x_1}{x_1 + i_g}, 1, 1\right).$$

Then we get the closed form pricing formula

$$A_t^1 = NPe^{-(T-t)f_{t,T}} \left(\sum_{i=1}^{\min[t,m]} \prod_{j=i}^{t} v_j \prod_{j=t+1}^{T} w_j + \sum_{i=t+1}^{m} \prod_{j=i}^{T} w_j \right). \tag{5}$$

A proof follows from Appendix 1 in Nonnenmacher (1998b).

For product 2, however, no explicit pricing formulas are available. Hence, calculating A_t^2 requires numerical techniques.

For all calculations in the following Sections, we use interest rate and volatility data as of December 18, 1997. We use a volatility of $\sigma = 23.92\%$, as quoted in the December 21 issue of Handelsblatt, and the term structure of interest rates (in %) as given in table 1:

t	1	2	3	4	5	6	7	8	9	10	11	12
$f_{0,t}$	3.93	4.41	4.69	4.89	5.07	5.23	5.36	5.48	5.57	5.66	5.71	5.76

Table 1: The term structure of interest rates

	Product 1	Product 2
$\Delta r = 0,\ \Delta\sigma = 0$	39.2	158.9
$\Delta r = 1\%,\ \Delta\sigma = 0$	45.3	174.6
$\Delta r = 0,\ \Delta\sigma = 2\%$	36.9	153.0

Table 2: The rates of index participation

5 The Rate of Index Participation

Given the yield curve, the index volatility and $i_g \geq 0$, the fair rate of index participation for our products can now be calculated as a solution $x > 0$ to the implicit equation

$$A_0 = A_0^k(x_k) = \sum_{i=0}^{m-1} NPe^{-\int_0^i r(s)ds} = \text{present value of net premiums,} \tag{6}$$

if such a solution exists. Hence, the fair rate of index participation is that value of x, for which the price at time 0 of a security that pays of the benefit at expiration equals the present value at time 0 of the net premiums. For both products ($k = 1, 2$) it is true, that there exists a unique solution $x > 0$ of (6) if and only if $i_g < i^*$ for some i^* that depends only on the given yield curve (but not on k or σ). In Nonnenmacher (1998a) it is shown how in equation (6) the costs for hedging against negative market values of the policy have to be taken into account. Within our model and given current market data, the impact of these costs is negligible, cf. Nonnenmacher and Ruß (1999). Therefore, we here only consider equation (6) for simplicity. This justifies the assumption made in Section 2.

For given market data, the rate of index participation for product 1 can be calculated by simple numerical methods from (6) using the explicit pricing formula presented in Section 4. For product 2, however, in every iteration of such a numerical method, $A_0^2(x_2)$ has to be calculated using Monte Carlo simulation techniques.

Table 2 shows the rate of index participation for both products and different market scenarios. Here, Δr means a parallel shift of the term structure of interest rates by Δr, and $\Delta\sigma$ denotes a change of the volatility by $\Delta\sigma$. We fix $i_g = 2\%$ for both products which approximately equals the current rate of inflation in Germany.

We can see that for both products, the rate of index participation is increasing in Δr and decreasing in $\Delta\sigma$. This holds in general as the guarantee becomes cheaper when interest rates rise and on the other hand, the included options become chaper if the volatility is reduced. Furthermore, the rate of index participation of product 1 is rather low since here a cliquet version with compound interest is used. Since product 2 is linked to the averaging return of the index, the fair participation rate is much higher, cf. also Ruß and Schittenhelm (1998) where we use a great variety of 42 different payoff functions to show the impact of certain product features on the fair rate of index participation.

6 Calculation of the Risk Premium

6.1 Marked to market calculation of the Risk Premium

The m risk premiums RP are paid at $t = 0, \ldots, m-1$. We assume in what follows, that if the insured person dies in $(t-1, t]$, the sum payable at death D_t is paid at time t. As $D_t = D_t^e + V_t$ and V_t results from the savings process, the risk premium is a solution to the implicit equation

$$RP \sum_{k=0}^{m-1} E_Q \left[\chi_{\{\tau > k\}} e^{-\int_0^k r(s)ds} \right] = \sum_{k=0}^{T-1} E_Q \left[\chi_{\{k < \tau \leq k+1\}} e^{-\int_0^{k+1} r(s)ds} D_{k+1}^e \right], \tag{7}$$

where the random variable τ denotes the time of death of the insured person and χ is the indicator function. Assuming the financial market to be independent of the mortality of the insured person, the insurance company to be risk neutral with respect to mortality and the insured person to be of age z at the time the policy is sold, (7) becomes (cf. Aase and Persson (1994))

$$RP \sum_{k=0}^{m-1} {}_kp_z e^{-\int_0^k r(s)ds} = \sum_{k=0}^{T-1} {}_kp_z \; q_{z+k} E_Q \left[e^{-\int_0^{k+1} r(s)ds} D_{k+1}^e \right], \tag{8}$$

where q_ζ denotes the probability that an insured person of age ζ dies within the next year and ${}_kp_\zeta = \prod_{i=0}^{k-1}(1 - q_{\zeta+i})$ is the probability that an insured person of age ζ survives the next k years. In all the following calculations, we use the mortality table DAV 1994 T, men. This is the mortality table of the German Society of Actuaries.

Note that the risk premium cannot be calculated directly from (8) since the D_{k+1}^e depend on the unknown gross premium GP and therefore on RP.

6.2 Traditional calculation of the Risk Premium

If the D_{k+1}^e are deterministic (as in our risk model 1) the traditional actuarial way of calculating the risk premium can also be applied. This procedure leads to an equation similar to (8) but with a constant discrete, annualised discount rate R. According to German legislation, $R \leq 4\%$ is required for calculation of policy reserves. German insurance companies usually always use $R = 4\%$ for the premium calculation, as well, regardless of the current term structure of interest rate.

$$RP \sum_{k=0}^{m-1} {}_kp_z (1+R)^{-k} = \sum_{k=0}^{T-1} {}_kp_z \; q_{z+k} D_{k+1}^e (1+R)^{-(k+1)}. \tag{9}$$

7 Empirical Results

7.1 A model with explicit costs

We first fix the costs for our insurance contract, since they are part of the gross premium.

We assume acquisition costs of $\alpha m GP$, payable at $t = 0$ and depending on the sum of the gross premiums. Collection costs, payable at time $t = 0, 1, \ldots, m-1$ are assumed to be βGP. Mangement fees amounting to $\gamma m GP$ and a fixed sum of C for each policy are also payable at $t = 0, 1, \ldots, m-1$. Hence, within our model, the (constant) gross premium is a solution of

$$(GP - NP) \sum_{k=0}^{m-1} {}_kp_z e^{-\int_0^k r(s)ds} = (RP + (\beta + \gamma m) GP + C) \sum_{k=0}^{m-1} {}_kp_z e^{-\int_0^k r(s)ds} + \alpha m GP$$

or, equivalently,

$$GP = \frac{NP + RP + C}{(1 - \beta - \gamma m) \sum_{k=0}^{m-1} {}_kp_z e^{-\int_0^k r(s)ds} - \alpha m} \sum_{k=0}^{m-1} {}_kp_z e^{-\int_0^k r(s)ds}. \tag{10}$$

Note that (10) is an implicit equation, as RP depends on GP, cf. (8).

By simple iterative numerical methods like the bisection or secant method, GP is calculated from (10). In every iteration, however, we have to calculate RP from (8) which, in the case of risk model 2 and 3 requires Monte Carlo simulation methods; see Appendix A for details. The reader should note that for risk model 1, the risk and gross premiums are the same for both products.

When calculating the gross premium the traditional way, the constant discount rate R is used in (10). This leads to

$$GP = \frac{NP + RP + C}{(1 - \beta - \gamma m) \sum_{k=0}^{m-1} {}_kp_z (1+R)^{-k} - \alpha m} \sum_{k=0}^{m-1} {}_kp_z (1+R)^{-k}. \tag{11}$$

Product 1 or 2, Risk Model 1						
	Risk Premium			Gross Premium		
	$z = 20$	$z = 40$	$z = 60$	$z = 20$	$z = 40$	$z = 60$
$\Delta r = \Delta\sigma = 0\%$	179.50	523.05	3632.20	21587.02	21956.18	25311.21
$\Delta r = 1\%,\ \Delta\sigma = 0\%$	172.85	497.65	3444.39	21599.22	21948.71	25133.32
$\Delta r = 0\%,\ \Delta\sigma = 2\%$	179.50	523.05	3632.20	21587.02	21956.18	25311.21
traditional	192.88	576.37	4028.07	21587.84	21999.35	25718.07

Table 3: Risk and Gross Premium for Risk Model 1

Product 1, Risk Model 2						
	Risk Premium			Gross Premium		
	$z = 20$	$z = 40$	$z = 60$	$z = 20$	$z = 40$	$z = 60$
$\Delta r = \Delta\sigma = 0\%$	33.75	82.83	534.73	21431.55	21486.53	22001.38
$\Delta r = 1\%,\ \Delta\sigma = 0\%$	33.66	82.63	533.13	21450.59	21505.52	22018.76
$\Delta r = 0\%,\ \Delta\sigma = 2\%$	33.75	82.83	534.68	21431.51	21486.51	22001.35

Table 4: Risk and Gross Premium for Product 1, Risk Model 2

Tables 3 - 7 show the gross and risk premiums for our two products and the three risk models under different market scenarios and different ages of the insured person (assumed to be male). Furthermore, the gross and risk premiums in model 1 that result from the traditional way are stated. In our calculation, we used the rates of index particitation calculated in Section 5 and we let $\alpha = 4\%,\ \beta = 1.25\%\ \gamma = 0.125\%\ C = 55DEM,\ NP = 20000DEM,\ i_g = 2\%,\ m = 5$ and $T = 12$ in both products.

An interpretation of the above data is given in the following Subsection.

7.2 A model allowing for implicit costs

In the results of the previous Subsection, all costs were explicitly added to the net premium. In what follows, we discuss an alternative approach that allows us to implicitly calculate some (or all) costs. For example, we look at the case where the complete acquisition costs are deducted up front from the first net premium leading to a reduced rate

Product 1, Risk Model 3						
	Risk Premium			Gross Premium		
	$z = 20$	$z = 40$	$z = 60$	$z = 20$	$z = 40$	$z = 60$
$\Delta r = \Delta\sigma = 0\%$	22.92	44.38	312.79	21419.98	21445.49	21764.91
$\Delta r = 1\%,\ \Delta\sigma = 0\%$	22.66	43.68	307.98	21438.84	21463.93	21779.33
$\Delta r = 0\%,\ \Delta\sigma = 2\%$	22.92	44.38	313.29	21419.97	21445.84	21764.85

Table 5: Risk and Gross Premium for Product 1, Risk Model 3

Product 2, Risk Model 2						
	Risk Premium			Gross Premium		
	$z=20$	$z=40$	$z=60$	$z=20$	$z=40$	$z=60$
$\Delta r=\Delta\sigma=0\%$	35.03	86.36	546.87	21433.15	21490.61	22020.56
$\Delta r=1\%,\ \Delta\sigma=0\%$	35.09	86.77	559.20	21451.94	21506.79	22040.46
$\Delta r=0\%,\ \Delta\sigma=2\%$	34.59	85.88	565.07	21433.28	21489.98	22017.17

Table 6: Risk and Gross Premium for Product 2, Risk Model 2

Product 2, Risk Model 3						
	Risk Premium			Gross Premium		
	$z=20$	$z=40$	$z=60$	$z=20$	$z=40$	$z=60$
$\Delta r=\Delta\sigma=0\%$	24.30	48.23	338.24	21421.67	21449.11	21785.14
$\Delta r=1\%,\ \Delta\sigma=0\%$	24.30	45.34	334.46	21440.46	21469.04	21807.99
$\Delta r=0\%,\ \Delta\sigma=2\%$	24.99	48.17	329.52	21423.06	21449.26	21793.33

Table 7: Risk and Gross Premium for Product 2, Risk Model 3

of index participation which is then calculated from the equation

$$A_0 = A_0^k(x_k) = \sum_{i=0}^{m-1} NPe^{-\int_0^i r(s)ds} - \hat{C}, \tag{12}$$

where $\hat{C}$ denotes the acquisition costs. The same can be done for other costs, analogously. The reduced rates of index participation for $\hat{C} = 4000\ DEM$ are denoted by $\hat{x}$ and given in Table 8. For comparison, we also give the rates of index participation x that have been calculated from (6). The reader should note that the design of the contract does not change. Only the acquisition costs are now hidden in the rate of index participation.

We now calculate the risk and gross premiums letting $\alpha = 0$ and get results given in tables 9 - 13.

We can see from the results, that (as expected) risk model 1 leads to the highest risk (and hence gross) premium while the values for model 2 exceed those for model 3. In model 1, the traditional way of calculating the risk premium leads to a higher result since the average level of the term structure of interest rates used here is higher than $R = 4\%$.

	Product 1		Product 2	
	$\hat{x}$ (in%)	x (in%)	$\hat{x}$ (in%)	x (in%)
$\Delta r=\Delta\sigma=0\%$	35.6	39.2	142.0	158.9
$\Delta r=1\%,\ \Delta\sigma=0\%$	41.8	45.3	158.2	174.6
$\Delta r=0\%,\ \Delta\sigma=2\%$	33.5	36.9	137.0	153.0

Table 8: Reduced rates of index participation

Product 1 or 2, Risk Model 1						
	Risk Premium			Gross Premium		
	$z = 20$	$z = 40$	$z = 60$	$z = 20$	$z = 40$	$z = 60$
$\Delta r = \Delta\sigma = 0\%$	171.40	499.00	3435.31	20612.89	20946.76	23939.17
$\Delta r = 1\%,\ \Delta\sigma = 0\%$	164.90	474.36	3255.65	20606.27	20912.64	23756.08
$\Delta r = 0\%,\ \Delta\sigma = 2\%$	171.40	499.00	3435.31	20612.89	20946.76	23939.17
traditional	184.29	550.16	3809.12	20626.03	20998.89	24320.12

Table 9: Risk and Gross Premium for Risk Model 1 in a model allowing for implicit costs

Product 1, Risk Model 2						
	Risk Premium			Gross Premium		
	$z = 20$	$z = 40$	$z = 60$	$z = 20$	$z = 40$	$z = 60$
$\Delta r = \Delta\sigma = 0\%$	34.01	82.37	533.66	20472.90	20522.17	20982.77
$\Delta r = 1\%,\ \Delta\sigma = 0\%$	33.90	82.11	531.44	20472.80	20521.92	20980.81
$\Delta r = 0\%,\ \Delta\sigma = 2\%$	34.03	82.44	533.72	20472.91	20522.21	20982.96

Table 10: Risk and Gross Premium for Product 1, Risk Model 2 in a model allowing for implicit costs

Product 1, Risk Model 3						
	Risk Premium			Gross Premium		
	$z = 20$	$z = 40$	$z = 60$	$z = 20$	$z = 40$	$z = 60$
$\Delta r = \Delta\sigma = 0\%$	23.93	46.29	326.10	20462.62	20485.43	20770.34
$\Delta r = 1\%,\ \Delta\sigma = 0\%$	23.65	45.53	320.58	20462.34	20484.63	20765.85
$\Delta r = 0\%,\ \Delta\sigma = 2\%$	23.95	46.34	325.86	20462.63	20485.47	20770.23

Table 11: Risk and Gross Premium for Product 1, Risk Model 3 in a model allowing for implicit costs

Product 2, Risk Model 2						
	Risk Premium			Gross Premium		
	$z = 20$	$z = 40$	$z = 60$	$z = 20$	$z = 40$	$z = 60$
$\Delta r = \Delta\sigma = 0\%$	35.50	84.20	546.33	20473.97	20524.30	21001.92
$\Delta r = 1\%,\ \Delta\sigma = 0\%$	35.54	83.46	547.76	20473.65	20523.91	21003.75
$\Delta r = 0\%,\ \Delta\sigma = 2\%$	35.19	85.14	550.19	20474.51	20523.36	20998.58

Table 12: Risk and Gross Premium for Product 2, Risk Model 2 in a model allowing for implicit costs

Product 2, Risk Model 3						
	Risk Premium			Gross Premium		
	$z=20$	$z=40$	$z=60$	$z=20$	$z=40$	$z=60$
$\Delta r = \Delta\sigma = 0\%$	25.44	48.02	341.19	20463.58	20488.73	20791.94
$\Delta r = 1\%,\ \Delta\sigma = 0\%$	25.53	47.17	339.29	20463.74	20487.19	20785.71
$\Delta r = 0\%,\ \Delta\sigma = 2\%$	25.14	47.58	347.79	20464.07	20489.35	20785.06

Table 13: Risk and Gross Premium for Product 2, Risk Model 3 in a model allowing for implicit costs

Nevertheless, if interest rates further decline, it might happen, that the traditional risk premium falls below the marked to market risk premium leading to a systematic loss.

Furthermore, the premiums in model 1 are the same for both products, as the effective sum payable at death does not depend on the payoff function A_T.

The sensitivities of the risk and gross premium with respect to changes in the term structure of interest rates and index volatility are quite low. Furthermore, the results for the two different products are very similar while the premiums for product 2 slightly exceed those for product 1. As expected, we further see a rather high sensitivity with respect to age.

The deduction of up front costs leads to a rather small reduction of the rate of index participation and the differences in the risk premiums are rather small, as well. Only for risk model 1, the differences in the risk premium are higher. This is easily understood since the participation rate does not enter the corresponding equations and hence the only difference is the value of α. The gross premium is generally reduced by about 1000 DEM as expected. Therefore, from a marketing point of view, a deduction of up front costs might be desirable, in particular, to hide acquisition costs.

8 Summary and Outlook for Further Research

In a series of former papers, we dealt with the financial aspects of guaranteed equity-linked life insurance contracts in Germany. This, in particular, requires the calculation of today's fair price of the payoff structure promised in case the policy holder survives the term of the contract.

In this paper, we addressed the question of how to price that part of the sum payable at death that exceeds the market value of the pure investment part for two different equity-linked products, which have been sold in the German and Swiss market, respectively. We analysed three popular payoff profiles for the death benefits and used the arbitrage pricing theory in a deterministic interest rate environment in order to calculate the associated risk premiums under different market scenarios. Also, we looked at the gross premiums, now including acquisition and management fees since this premium is finally charged to the policyholder. For comparison, we also calculated the premiums using traditional actuarial methods if applicable.

The main results of this paper are twofold: First, since in general the sum payable at death is stochastic, modern finance theory must be employed in order to determine today's market price. Using traditional actuarial methods, which can only be applied in case of non-random quantities, leads to a bias resulting in a systematic profit or loss. This also implies that the life insurance company is now exposed to an additional financial risk which has to be analysed and hedged. Second, the main costs associated with a life insurance policy, the acquisition costs can be hidden in the index participation rate of an equity-linked contract by subtracting those from the first net premium. Our analysis shows that the corresponding reduction of the participation rate is minor, but the gross premium is significantly reduced. This should be kept in mind, when the policy is marketed.

In order to further reduce the gross premium, a minimum sum payable at death should be chosen, which is still consistent with German legislation.

Our results show a moderate sensitivity of the calculated premiums with respect to the level of interest rate. Although the market value of the investment part of the policy depends heavily on the term structure of interest rates, this is not surprising, since the dependency is reflected in the rate of index participation, cf. equation (6). However, it seems to be worthwile to carry out our anaysis in a stochastic interest rate economy, in particular, when the involved options are taken into account.

Furthermore, we see that the risk premiums depend in either case heavily on the age of the insured person. This leads to a problem if the policy is calculated using an average age.

A Our Simulation Approach

A.1 Determination of the rate of index participation

To determine the rate of index participation, we have to calculate a solution of the implicit equation (6). If however up front costs are deducted, we use equation (12) and proceed as described in the rest of this Section. This is achieved by simple numerical procedures like bisection or secant methods. In every step, we have to calculate the value A_0 which in case of product 1 is done by our explicit pricing formula (5). Hence, no Monte Carlo simulation techniques are required. In the case of product 2, however, in every iterative step, we have to calculate A_0 by Monte Calo simulation. We create 10000 DAX30 simulation paths according to (4). For each simulation path, we calculate the discounted payoff $e^{-\int_0^T r(s)ds} A_T$ and the arithmetic average of these discounted payoffs is our Monte Carlo estimate for A_0.

A.2 Determination of the risk and gross premium

In the case of risk model 1, we do not need any Monte Carlo simulation techniques. Instead, we calculate the risk and gross premium directly from (8), (10) and (9), (11), respectively in an iterative process similar to the one described below. Note that in this model, the values D_{k+1}^e are deterministic.

In risk models 2 or 3, we determine the gross premium by iterating equation (10). We start with $GP_0 = NP$ and then, for given GP_ν, we calculate $GP_{\nu+1}$ from (10), having determined RP_ν from (8) using GP_ν to calculate the values D_{k+1}^e. For given k, the determination of $E_Q[e^{-\int_0^{k+1} r(s)ds} D_{k+1}^e]$ requires Monte Carlo simulation techniques. In a first level of simulation, we generate 10000 simulation paths for the DAX30 from 0 to $k+1$ according to (4). For each of these simulation paths, we calculate A_{k+1}. In the case of product 1 this is done using our explicit pricing formula and in the case of product 2 we use a second level of 10000 simulation paths for the DAX30 from $k+2$ to T, again according to (4). Each of these level 2 simulation paths yields a discounted payoff $e^{-\int_{k+1}^{T} r(s)ds} A_T$. The average of these discounted payoffs is again our estimate for the value A_{k+1}. For every level 1 simulation we therefore get a value of A_{k+1} from which we calculate V_{k+1} and hence (using GP_ν in iteration number ν) D_{k+1}^e. Discounting these values and again taking the average yields our estimate for $E_Q[e^{-\int_0^{k+1} r(s)ds} D_{k+1}^e]$. With these values, RP_ν can be deducted from (8) and hence $GP_{\nu+1}$ can be calculated from (10).

This procedure is iterated until $\mid GP_{\nu+1} - GP_\nu \mid \leq \epsilon$. All the results in Section 7 were calculated letting $\epsilon = 0.1 DEM$.

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