GMWB For Life

An Analysis of Lifelong Withdrawal Guarantees

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Abstract

We analyze the latest guarantee feature in the variable annuities market: guaranteed minimum withdrawal benefits for life or guaranteed lifelong withdrawal benefits. This option gives the client the right to deduct a certain amount annually from the policy's account value until death – even if a unit-linked account value drops to zero. We show how such products can be analyzed within a general framework presented in Bauer et al. (2006). We price the embedded guarantee for different product designs and parameters under deterministic and optimal client behaviour.

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1 Introduction

Variable annuities, i.e. deferred annuities that are fund-linked during the accumulation period were introduced in the 1970s in the United States (see Sloane (1970)). Since the 1990s, certain optional guarantees are usually offered in such policies: guaranteed minimum death benefits (GMDB) as well as guaranteed minimum living benefits (GMLB). The GLWB options can be categorized in three main groups: Guaranteed minimum accumulation benefits (GMAB) provide a guaranteed minimum survival benefit at some or several specified points of time, guaranteed minimum income benefits (GMIB) offer a guaranteed lifelong fixed annuity starting at the end of the deferment period. However, GMIB products offer no guaranteed lump sum benefit. Finally, in guaranteed minimum withdrawal benefits (GMWB) some specified amount is guaranteed for withdrawals during the life of the contract as long as both the amount that is withdrawn within each policy year and the total amount that is withdrawn over the term of the policy stay within certain limits. A detailed product description and extensive literature overview is given in Bauer et al. (2006) and is therefore omitted in this paper.

In the present paper, we look at the latest variant of GMWB-guarantees, so called GMWB for life. Such products have recently been introduced in the US, Asia and Europe. As the name suggests, GMWB for life or Guaranteed Lifelong Withdrawal Benefits (GLWB) offer a lifelong withdrawal guarantee. Therefore, there is no limit for the total amount that is withdrawn over the term of the policy. Usually, with an immediate GLWB, annually a certain percentage of the single premium (the guaranteed amount) can be withdrawn from the policy. This percentage rate may depend on the age of the insured. If at the time of death there is a remaining account value, then this value is paid to the beneficiary as death benefit. If, however, due to declining stock markets and/or such withdrawals the account value of the policy becomes zero while the insured is still alive, then the insured can continue to withdraw the guaranteed amount annually until death. The insurer charges a fee for this guarantee which is usually a fix percentage of the policyholder's account value per annum. In deferred versions of the product, the product is fund linked with or without guarantee during the deferment period. The account value at the end of the deferment period (or some guaranteed amount if guarantees are included in the deferment period as well) is then treated like a single premium to an immediate GLWB contract.

The rest of the paper is organized as follows. In Section 2, we give a detailed description of GLWB products. Section 3 introduces our model. Here, we build on a general model introduced in Bauer et al. (2006) and show, how GLWB products can be included in their model. Due to the complexity of the products, in general there are no closed form solutions for the valuation problem. Therefore, we have to rely on numerical methods and present our valuation approach in Section 4. We show how a given contract can be priced with Monte Carlo methods assuming deterministic as well as "optimal" policyholder behavior. For the latter, we introduce numerical methods for the determination of such optimal strategies. Finally, in Section 5 we present the results of our analyses that are derived in a Monte Carlo framework. We give the fair value of the guarantee for a variety of contracts, analyze the influence of several parameters and give economic interpretations. Section 6 closes with a summary of the main results and an outlook for future research.

2 Guaranteed Minimum Withdrawal Benefits for Life

In this Section, we explain the concept of GLWB within variable annuities. We start with a very brief description of variable annuities in general and other types of guarantees offered within such products and refer the reader to Bauer et al. (2006) for more details.

Variable Annuities are deferred, fund-linked annuity contracts, usually with a single premium payment up-front. Therefore, in what follows we restrict ourselves to single premium policies. When concluding the contract, the insured are frequently offered optional guarantees, which are paid for by additional fees.

The single premium P is invested in one or several mutual funds. We call the value A_t of the insured's individual portfolio the insured's account value. Customers can usually influence the risk-return profile of their investment by choosing from a selection of different mutual funds. All fees are taken from this account by cancellation of fund units. Furthermore, the insured has the possibility to surrender the contract, to withdraw a portion of the account value (partial surrender), or to annuitize the account value after a minimum term.

If the insured dies during the deferment period, the beneficiary obtains a death benefit that depends on the account value. If a guaranteed minimum death benefit is chosen, then the death benefit paid is the higher of the account value and some specified guaranteed value. Since the late 1990s, guaranteed minimum living benefits are offered in the market. The two earliest forms, *guaranteed minimum accumulation benefits* (GMAB) and *guaranteed minimum income benefits* (GMIB) offer the insured a guaranteed maturity benefit, i.e. a minimum benefit at the maturity T of the contract (or within a certain period of time). However, with the GMIB, this guarantee only applies if the account value is annuitized.

Since 2002, a new form of GLWB is offered: so-called *guaranteed minimum withdrawal benefits* (GMWB). Products with a GMWB option give the policyholder the possibility to withdraw a specified amount G_0^W (that usually coincides with the single premium) in small portions. Typically, the insured is entitled to annually withdraw a certain portion x_W of this amount G_0^W, even if the account value has fallen to zero, until the total withdrawals reach G_0^W. At maturity, the policyholder can take out or annuitize any remaining funds if the account value did not vanish due to withdrawals. These guarantees are extremely popular. In the first half of 2005, more than 3 out of 4 variable annuity contracts sold included a GMWB option. Each of the 15 largest variable annuity providers offered this kind of guarantee at that time (cf. Lehmann Brothers (2005)). Since GMWB products are so popular, insurers developed several versions of this product to be more competitive.

The latest version that is now already offered in the USA, Asia and Europe includes a lifelong withdrawal guarantee: The total amount to be withdrawn G_0^W is not limited. Insurers that introduced this type of guarantee (GLWB) were immediately able to significantly increase their new business volume. Usually, the annual amount to be withdrawn in a GLWB product is a certain percentage x_{WL} of the single premium P. Any remaining account value at the time of death is paid to the beneficiary as death benefit. If, however, the account value of the policy drops to zero while the insured is still alive, the insured can still continue to withdraw the guaranteed amount annually. The insurer charges a fee for this guarantee which is usually a pre-specified annual percentage of the account value. Deferred versions of the product also exist in the market. Here, the product is fund linked during the deferment period. The account value at the end of the deferment period is then treated like a single premium to an immediate GLWB contract. Sometimes, certain guarantees are also included in the deferment period: if the account value at the end of the deferment period is lower than some guaranteed amount, than this guaranteed amount is transferred to the GLWB payout phase.

Furthermore, different additional features like *step-ups* and *roll-ups* are offered with GLWBs: If a step-up is included, the guaranteed annual withdrawal amount can be increased at pre-specified points in time. At these step-up dates, the guaranteed annual withdrawal amount is increased if the portion x_{WL} of the current account value exceeds the previous guaranteed annual withdrawal amount.[1] Therefore, step-ups only occur if the policyholder's funds yield high performance and the account value has not been decreased heavily due to previous withdrawals. Common step-up features are, e.g., annual ratchet guarantees.

If a roll-up is included within GLWB, the annual guaranteed withdrawal amount is increased by a fixed percentage every year during a certain time period but only if the policyholder has not started withdrawing money. Therefore, roll-ups are commonly used as an incentive to the policyholder not to withdraw money from the account in the first years.

3 The Model

3.1 The Financial Market

We assume that there exists a probability space (Ω, F, Q) with a filtration $\boldsymbol{F} = (\Im_t)_{t \in [0,T]}$ and a risk neutral probability measure Q. Under this risk-neutral measure, payment streams can be valued as expected discounted values using the risk-neutral valuation formula (cf. e.g. Bingham and Kiesel (2004)). Existence of this measure also implies an arbitrage free financial market. Thus, for every regular derivative, there exists some self-financing investment strategy which replicates the payoff of the derivative. This allows the insurer to hedge the liabilities.

We assume a numéraire process $(B_t)_{t \in [0,T]}$ which evolves according to

$$\frac{dB_t}{B_t} = r_t dt, \; B_0 > 0, \tag{1}$$

where r_t denotes the short rate of interest at time t. For all our numerical calculations, we assume $r_t = r$. Thus, for the bank account we have $B_t = e^{rt}$.

We further assume the existence of some risky asset S_t that serves as underlying mutual fund for the considered variable annuity contracts. S_t evolves according to a Geometric Brownian motion with constant coefficients under Q, i.e.

[1] Typically, these step-up dates are annually or every three or five years on the policy anniversary date.

$$\frac{dS_t}{S_t} = rdt + \sigma dZ_t, \; S_0 = 1, \tag{2}$$

where σ denotes the (constant) volatility of the risky asset. Under the risk-neutral probability measure, the discounted asset process $\left(\frac{S_t}{B_t}\right)_{t\in[0,T]}$ is a martingale. At t = 0, we assume $S_0 = B_0 = 1$.

3.2 A Model for the Insurance Contract

In Bauer et al. (2006), a general model for the description and valuation of variable annuity contracts was introduced. Within this framework, any contract with one or several living benefit guarantees and/or a guaranteed minimum death benefit can be represented. In the numerical analysis however, only contracts with a rather short finite time horizon were considered. In particular, GLWB were not analyzed. In what follows, we therefore describe how GLWB-products can be included in this model. We refer to Bauer et al. (2006) for the explanation of other living benefit guarantees and more details on the model.

Let

- x_0 be the insured's age at the start of the contract,
- ${}_t p_{x_0}$ be the probability for a x_0-year old to survive the next t years,
- q_{x_0+t} be the probability for a $(x_0 + t)$-year old to die within the next year, and
- ω be the limiting age of the mortality table, i.e. the age beyond which survival is impossible.

The probability that an insured aged x_0 at inception passes away in the year $(t,t+1]$ is thus given by ${}_t p_{x_0} \cdot q_{x_0+t}$. The limiting age ω allows for a finite time horizon $T = \omega - x_0$.

We denote the single premium invested into the contract at time $t = 0$ by P. A_t denotes the account value at time t. Throughout the paper, we assume that all fees and charges are deducted continuously as a percentage φ of the account value and no upfront charges exist. This leads to $A_0 = P$. Besides, we allow for a surrender fee s which is charged as a percentage of any withdrawal of funds exceeding guaranteed withdrawals within a GMWB or GLWB option.

Following the notation in Bauer et al. (2006), we denote the value of some *withdrawal account* at time t by W_t. Withdrawals up to time t are credited to this account and compounded with the risk-free rate of interest. At inception, we have $W_0 = 0$. Similarly, death benefits paid up to time t are accumulated in the *death benefit account* D_t which is also compounded by the risk-free rate until time T. At time t = 0, we have $D_0 = 0$.

Throughout this paper, we focus on guaranteed minimum withdrawal benefits and, in addition, allow for guaranteed minimum death benefits. We use the following notations again following Bauer et al. (2006): If the contract includes a GMDB option, the death benefit at time t is given by the greater of the current account value and the guaranteed minimum death benefit base G_t^D, i.e. $\max\{A_t; G_t^D\}$. If a GMDB is included, the initial amount of the guaranteed minimum death benefit base is given by $G_0^D = A_0$ if not stated otherwise. Contracts without a GMDB rider simply pay out the current fund net asset value in the case of death. Such contracts are modeled by letting $G_t^D = 0$ for all t.

For the modeling of GMWB or GLWB options, we introduce two processes G_t^W and G_t^E. We call G_t^W the total amount guaranteed for future withdrawals. For standard GMWB options, we usually have $G_0^W = A_0$ at inception. For GLWB, the total amount of withdrawals is unlimited and thus $G_t^W = \infty$ for all t. The maximum amount that may be withdrawn annually due to the GMWB or GLWB-option is called G_t^E. At $t = 0$ it is given by some percentage of the single premium P, i.e. $G_0^E = x_W A_0$. Please note that we use the same state variables for the modeling of GMWB and GLWB options since we do not consider contracts with both, a GMWB and a GLWB-option. Such contracts could also be modeled in our framework by introducing separate processes G_t^W and G_t^E for each option.[2]

Finally, we call $y_t = \left(A_t, W_t, D_t, G_t^D, G_t^W, G_t^E\right)$ the state vector at time t containing all information about the contract at that point in time.

Since we restrict our analyses to single premium contracts and do not allow for additional premium payments, policyholder actions during the life of the contract are limited to withdrawals. Depending on the amount of money withdrawn, the policyholder can withdraw funds as a guaranteed withdrawal of a GMWB/GLWB option, perform a partial surrender, i.e. withdraw more than the guaranteed withdrawal amount, or completely surrender the contract.

For the sake of simplicity, we allow for withdrawals at policy anniversaries only. Also, we assume that death benefits are paid out at policy anniversaries if the insured person has died during the previous year. Thus, the value of the state variables described above may have discontinuities at times $t = 1,2,\ldots,T$. Thus, at each policy anniversary, we have to distinguish between the value of the respective variable $(\cdot)_t^-$ immediately before and the value $(\cdot)_t^+$ after withdrawals and death benefit payments.

During the year, all processes are subject to capital market movements and may therefore also change between two policy anniversaries. In what follows, we describe the development between two policy anniversaries and the transition at policy anniversaries for different contract designs. From these, we are finally able to determine all benefits for any given policy holder strategy and any capital market path. This allows for the valuation of such contracts in a Monte-Carlo framework.

3.2.1 Development between two Policy Anniversaries

We assume that the annual guarantee fee φ is deducted from the policyholder's account value on a continuous basis. Thus, the development of the account value be-

[2] Since G^W is not needed for GLWB options, two processes G^E and one process G^W are sufficient.

tween two policy anniversaries is given by the development of the underlying fund after deduction of the guarantee fee, i.e.

$$A_{t+1}^{-} = A_t^{+} \frac{S_{t+1}}{S_t} \cdot e^{-\varphi} . \qquad (3)$$

As described earlier, the withdrawals and death benefits are compounded on the accounts W_t and D_t at the risk-free rate of interest r. This leads to

$$W_{t+1}^{-} = W_t^{+} e^{\int_t^{t+1} r_s ds} \text{ and } D_{t+1}^{-} = D_t^{+} e^{\int_t^{t+1} r_s ds} .$$

Throughout this paper, we use return of premium death benefits if a GMDB option is included. Thus, the value of the guaranteed minimum death benefit base is not changed between two policy anniversaries which leads to $G_{t+1}^{D/E/W-} = G_t^{D/E/W+}$.

Since withdrawals only occur at integer times t, the processes G_t^W and G_t^E only change during the year if some roll-up feature is included. The time horizon within which roll-ups occur is usually limited to a certain number of years t_{WL} from the beginning of the withdrawal period. Roll-ups only are applied if no withdrawals have been made so far. We denote the annual roll-up rate by i_W and let $G_{t+1}^{W-} = G_t^{W+}\left(1+i_W\right)$ and $G_{t+1}^{E-} = G_t^{E+}\left(1+i_W\right)$ if $W_t = 0$ and $t \le t_{WL}$. If no roll-up is included, $G_{t+1}^{W-} = G_t^{W+}$. Besides roll-ups, withdrawal guarantees are often equipped with so called step-up features. If a step-up is included, withdrawal guarantees can only be increased at the policy anniversaries. Thus, step-ups are described in the following section.

3.2.2 Transition at a Policy Anniversary *t*

At the policy anniversaries, we have to distinguish the following four cases:

<u>a) The insured has died within the previous year (*t*-1,*t*]</u>

If the insured person has died within the previous policy year, the death benefit is credited to the death benefit account D_t : $D_t^{+} = D_t^{-} + \max\{G_t^{D-}; A_t^{-}\}$. With the payment of the death benefit, the insurance contract matures. Thus, the policyholder's account value and all riders attached to it are terminated, i.e. $A_t^{+} = 0$, $G_t^{D+} = 0$, $G_t^{W+} = 0$, and $G_t^{E+} = 0$. Withdrawals that have been made earlier remain on the withdrawal account W_t: $W_t^{+} = W_t^{-}$.

<u>b) The insured has survived the previous policy year and does not withdraw any money from his account at time *t*</u>

If no death benefit is paid out to the policyholder and no withdrawals are made from the contract, the account value as well as the withdrawal and the death benefit account remain unchanged, i.e. $A_t^{+} = A_t^{-}$, $D_t^{+} = D_t^{-}$ and $W_t^{+} = W_t^{-}$. Also the guaranteed minimum death benefit doesn't change in this case which leads to $G_t^{D+} = G_t^{D-}$.

If the contract includes a withdrawal guarantee with step-up and t is a step-up point, the total amount available for future withdrawals is increased if the account value exceeds the current value of the guarantee account, i.e. $G_t^{W+} = \max\left\{G_t^{W-}; A_t^{+}\right\}$. Note that in the case of GLWB, this value remains unchanged (at infinity). The annual guaranteed withdrawal amount G_t^E may also be increased. It is given by $G_t^{E+} = \max\left\{G_t^{E-}; x_W \cdot A_t^+\right\}$.

c) The insured has survived the previous policy year and at the policy anniversary withdraws an amount within the limits of the withdrawal guarantee

If the insured has survived the past year, no death benefits are paid and therefore $D_t^+ = D_t^-$. Any withdrawal E_t below the guaranteed annual withdrawal amount G_t^{E-} and lower than the total withdrawal amount G_t^{W-} reduces the account value by the withdrawn amount. Of course, we do not allow for negative policyholder account values and thus get $A_t^+ = \max\left\{0; A_t^- - E_t\right\}$. The withdrawal account is increased by the amount withdrawn, i.e. $W_t^+ = W_t^- + E_t$.
In the case of a GMWB option, the remaining total withdrawal amount is reduced by the amount withdrawn, i.e. $G_t^{W+} = G_t^{W-} - E_t$. For GLWB-guarantees, the total amount of withdrawals is unlimited and thus remains unchanged $G_t^{W+} = G_t^{W-} = \infty$. For both riders, the maximal annual withdrawal amount $G_t^{E+} = G_t^{E-}$ remains unchanged. If, however, a step-up feature is included and t is a step-up point, the total amount available for future withdrawals can be increased as described in b), i.e. $G_t^{W+} = \max\left\{G_t^{W-}; A_t^+\right\}$ (for GMWB only) and the new annual guaranteed withdrawal amount G_t^E is given by $G_t^{E+} = \max\left\{G_t^{E-}; x_W \cdot A_t^+\right\}$.

With any withdrawals, the guaranteed death benefits are reduced at the same rate as the account value, i.e. $G_t^{D+} = \left(\frac{A_t^+}{A_t^-}\right) G_t^{D-}$.

d) The insured has survived the previous policy year and at the policy anniversary withdraws an amount exceeding the limits of the withdrawal guarantee

In this case again, no death benefits are paid and therefore $D_t^+ = D_t^-$.

Withdrawals exceeding the limits of the withdrawal guarantee always lead to a partial or full surrender of the annuity contract, depending on the amount of money withdrawn and on the amount remaining within the policyholder's account.

Any withdrawal E_t exceeding the limits of the withdrawal guarantee can be separated into two parts, the guaranteed amount $E_{t+1}^1 = \min\left\{G_{t+1}^{E-}; G_{t+1}^{W-}\right\}$ and the exceeding part $E_t^2 = E_t - E_t^1$. As in case c), the account value is reduced by the amount withdrawn, i.e. $A_t^+ = A_t^- - E_t$, and the withdrawn amount is credited to the withdrawal account after deduction of surrender charges for the exceeding part.[3] Thus, we get $W_t^+ = W_t^- + E_t^1 + E_t^2 \cdot (1 - s)$.

[3] Note that surrender charges only apply to the exceeding part, Therefore surrender charges were not considered in case c) above.

In the case of a GMWB option, usually the remaining total withdrawal amount is reduced by the withdrawn amount, but at least by the same percentage, by which the account value is reduced.[4] From this we get $G_t^{W+} = \min\left\{G_t^{W-} - E_t; \frac{A_t^+}{A_t^-} G_t^{W-}\right\}$. For GLWB, the total amount of withdrawals is unlimited and thus again remains unchanged $G_t^{W+} = G_t^{W-} = \infty$. For both riders, the maximum annual withdrawal amount G_t^{E+} is reduced by the same percentage, by which the account value is reduced, i.e. $G_t^{E+} = \frac{A_t^+}{A_t^-} \cdot G_t^{E-}$. If a step-up feature is included and t is a step-up point, the total amount available for future withdrawals may be increased: $G_t^{E+} = \max\left\{\frac{A_t^+}{A_t^-} \cdot G_t^{E-}; x_W \cdot A_t^+\right\}$.

As in case c), with any withdrawals, the guaranteed death benefits are reduced at the same rate as the account value, i.e. $G_t^{D+} = \left(\frac{A_t^+}{A_t^-}\right) G_t^{D-}$.

3.2.3 Maturity Benefits at T

At maturity of the contract at $t = T = \omega - x$, the insured has either died or surrendered the contract. Thus, all insurance benefits have already been credited to D_t or W_t and no additional final payment is given to the policyholder. We therefore call W_T and D_T the maturity benefit of the contract.

4 Valuation of the Guarantees

Assuming independence between financial markets and mortality and risk-neutrality of the insurer with respect to mortality risk, we are able to use the product measure of the risk-neutral measure of the financial market and the mortality measure. In what follows, we denote this product measure by Q.

4.1 Deterministic Policyholder Behavior

If we assume deterministic policyholder behavior, any withdrawal strategy can easily be described by using a *withdrawal vector* $\bar{\xi} = (\xi_1; \ldots; \xi_T) \in \left(I\!R_+^\infty\right)^T$ [5] where ξ_t denotes the deterministic withdrawal amount at the end of year t, if the insured is still alive. Of course, if any such withdrawal exceeds the guaranteed annual withdrawal amount, the withdrawal leads to a partial or even full surrender. By allowing for $\xi_t = \infty$, a full surrender at time t can also be represented within such a strategy. Since deterministic strategies are already specified at time t = 0, every deterministic strategy is $\boldsymbol{F_0}$-measurable.

We denote the set of all possible $\boldsymbol{F_0}$-measurable strategies by $\Psi = \Psi_1 \times \ldots \times \Psi_T \subset \left(I\!R_+^\infty\right)^T$ .For any given strategy and under the assumption that the

[4] Whenever a non-guaranteed withdrawal occurs, future guarantees may be reduced. We here describe a so-called pro-rata reduction which is the predominant form in the market.

[5] Here, $I\!R_+$ denotes the non negative real numbers (including zero); furthermore we let $I\!R_+^\infty = I\!R_+ \cup \{\infty\}$.

insured dies in year $t \in \{1,2,...,\omega - x_0\}$, the maturity-values $W_T\left(t;\bar{\xi}\right)$ and $D_T\left(t;\bar{\xi}\right)$ are specified for each capital market path. Thus, the time zero value assuming the given policyholder strategy including all options is given by the risk-neutral discounted expected value of these maturity values:

$$V_0\left(\bar{\xi}\right) = \sum_{t=1}^{\omega - x_0} {}_{t-1}p_{x_0} \cdot q_{x_0+t-1} E_Q\left[e^{-\int_0^T r_s ds}\left(W_T\left(t;\bar{\xi}\right) + D_T\left(t;\bar{\xi}\right)\right)\right]. \quad (4)$$

4.2 Probabilistic Policyholder Behavior

If policyholders follow certain deterministic strategies with certain probabilities, we call this behavior probabilistic behavior. Policyholder strategies are still $\boldsymbol{F_0}$-measurable but several such strategies are now weighted by certain probabilities. We denote the corresponding deterministic strategies by $\bar{\xi}^{(j)} = \left(\xi_1^{(j)};...;\xi_T^{(j)}\right) \in \left(IR_+^{\infty}\right)^T$, $j = 1,2,...,n$ and the respective probabilities by $p_{\xi}^{(j)}$ where of course $\sum_{j=1}^{n} p_{\xi}^{(j)} = 1$. For any probabilistic strategy, the value of the contract under probabilistic policyholder behavior is given by

$$V_0 = \sum_{j=1}^{n} p_{\xi}^{(j)} V_0\left(\bar{\xi}^{(j)}\right). \quad (5)$$

4.3 Stochastic Policyholder Behavior

We call a policyholder strategy stochastic if the decision whether and how much money should be withdrawn depends on the account value or other information available at time *t*. Thus, stochastic policyholder strategies are not necessarily $\boldsymbol{F_0}$-measurable. However, we still assume some $\boldsymbol{F_t}$-measurable process **(*X*)**, which determines the amount to be withdrawn depending on the state vector y_t^- at time *t*. Thus, we get: $\boldsymbol{X}\left(t, y_t^-\right) = E_t$, $t = 1,2,...,T$.
Assuming that the insured dies in year $t \in \{1,2,...,\omega - x\}$, for each stochastic strategy **(*X*)** the values $W_T\left(t;\boldsymbol{(X)}\right)$ and $D_T\left(t;\boldsymbol{(X)}\right)$ are specified for any capital market path. Therefore, the value of the contract following some stochastic strategy **(*X*)** is given by:

$$V_0\left((\mathrm{X})\right) = \sum_{t=0}^{\omega - x_0} {}_{t-1}p_{x_0} \cdot q_{x_0+t-1} \cdot E_Q\left[e^{-\int_0^T r_s ds}\left(W_T\left(t,(\mathrm{X})\right) + D_T\left(t,(\mathrm{X})\right)\right)\right]. \quad (6)$$

We let Ξ denote the set of all admissible stochastic strategies. Then the value V_0 of a contract assuming a rational policyholder is given by

$$V_0 = \sup_{\boldsymbol{(X)} \in \Xi} V_0\left(\boldsymbol{(X)}\right). \quad (7)$$

Even though the value of the contract under rational policyholder behavior can easily be defined, the respective rational strategy is not obvious and cannot be easily determined. In the following section, we describe how Monte-Carlo simulation can be used to approximate optimal policyholder strategies. The ideas are based on Anderson (1999).

4.4 Determining the Contract Value using Monte Carlo Methods

By Itô's formula (see, e.g. Bingham and Kiesel (2004)), the iteration

$$A_{t+1}^{-} = A_t^{+} \frac{S_{t+1}}{S_t} \cdot e^{-\varphi} = A_t^{+} \cdot \exp\left\{ \left(r - \varphi - \frac{\sigma^2}{2} \right) + \sigma z_{t+1} \right\}; \quad z_{t+1} \sim N(0,1) \text{ iid},$$

can be conveniently used to produce realizations of sample paths $a^{(j)}$ of the policyholder's account using Monte Carlo Simulation.[6] For any capital market development and for any time of death, the evolution of all accounts and processes is determined by the rules given in Section 3.2. Thus, realizations of the benefits $w_T^{(j)}(t,(\mathrm{X})) + d_T^{(j)}(t,(\mathrm{X}))$ at time T are uniquely defined for any capital market scenario and the time zero value of these benefits in this sample scenario is given by

$$v_0^{(j)}((\mathrm{X})) = e^{-rT} \sum_{t=1}^{\omega - x_0} {}_{t-1}p_{x_0} \cdot q_{x_0+t-1} \left[w_T^{(j)}(t,(\mathrm{X})) + d_T^{(j)}(t,(\mathrm{X})) \right].$$

Hence, $V_0\left(\boldsymbol{(X)}\right) = \frac{1}{J}\sum_{j=1}^{J} v_0^{(j)}\left(\boldsymbol{(X)}\right)$ is a Monte-Carlo estimate for the value of the contract with J denoting the number of simulations.

At the end of each year the policyholder can decide what amount to withdraw from the account. In Milevsky and Salisbury (2006) the authors prove that within their model an optimal strategy for a GMWB contract can only be achieved by withdrawing either nothing or the guaranteed annual withdrawal amount or the total account value. In contrast to a GMWB, for a GLWB withdrawing nothing can never be optimal unless roll-ups or step-ups are included. Therefore in an optimal strategy, the policyholder can only withdraw the guaranteed amount or surrender the contract.

The property that optimal strategies can only be achieved by withdrawing the guaranteed annual withdrawal amount or the total account value also holds within our model: First, a withdrawal below the guaranteed annual withdrawal amount can never be optimal since no adjustments are made for future withdrawal guarantees in this case. Hence, when withdrawing less than the guaranteed annual withdrawal the future guarantees are the same than when withdrawing the full guaranteed amount. However, in the latter case, the account value is lower and thus the value of future withdrawal guarantees is higher. Furthermore, if a GMDB is included, the so-called additional death benefit, i.e. the part of the death benefit that exceeds the fund value, is reduced. Due to the martingale property of the underlying asset process and the guarantee fee that is deducted from the account value, the value of the additional death benefit is never greater than the withdrawal amount itself. Second, if it is optimal for the policyholder to withdraw more than the guaranteed annual withdrawal amount, than it has to be optimal to completely surrender the contract since all changes of the state variables occur on a pro-rata basis only.

In the case of surrender, the policyholder receives the account value after deduction of surrender charges; in return he waives all claims arising from the GLWB-option, i.e. the lifelong guaranteed annual withdrawals and future death or surrender benefits.

[6] For an introduction to Monte Carlo methods see, e.g., Glasserman (2003).

Thus, under optimal behaviour, the policyholder would withdraw exactly the annual guaranteed amount each year until the value of the underlying fund less surrender fee exceeds the expected value of future benefits. Then, he would surrender the contract.

We now describe how optimal policyholder strategies can be found using Monte-Carlo simulations. The task is to maximize a contract's value by surrendering at an optimal point of time allowing for $\mathbf{F_t}$-measurable strategies only.

We call $K = K_1 \times ... \times K_{\omega - x_0 - 1} \subseteq R^{\omega - x_0 - 2}$ an exercise strategy. Following this strategy, the contract is surrendered at time *t* if and only if the contract has not been surrendered before and $A_t^- \in K_t$, i.e.

$1_{K_t}(A_t^-) = 1$ and $1_{K_u}(A_u^-) = 0 \quad \forall u \in \{1,...,t-1\}$.

The value of the contract under some given strategy K is then given by $V_0(K)$ as defined above. For this given strategy, the contract is surrendered at the stopping time $\tau(K) := \inf\{t \in \{1,...,\omega - x_0 - 1\} \mid A_t^- \in K_t\}$ or not surrendered at all, i.e. $\tau(K) := \omega - x_0$, if $A_t^- \notin K_t \quad \forall t \in \{1,...,\omega - x_0 - 1\}$. This value can be easily calculated by Monte Carlo methods.

In what follows we explain how an optimal strategy can be approximated. Obviously, an optimal strategy has to be of the form $K = [k_1, \infty) x ... x [k_{\omega - x_0 - 1}, \infty) \subseteq R^{\omega - x_0 - 2}$: if it is optimal to surrender at some fund value, it has to be optimal to surrender at any higher fund value, as well, since a higher fund value leads to a higher surrender value if the contract is surrendered, and a lower value of future guarantees as well as a higher future guarantee fee if the contract is not surrendered. Analogously, if it is optimal not to surrender the contract at a given value, it is also optimal not to surrender at any lower fund value. For the determination of the optimal surrender boundaries $k_1,...,k_{\omega - x_0 - 1}$ we use the following backward induction algorithm:

First, we determine the optimal strategy at time $t = \omega - x_0 - 1$ by maximizing the contract value following a given strategy $[k_{\omega - x_0 - 1}, \infty)$ for $k_{\omega - x_0 - 1}$. Second, we determine some optimal strategy at $t = \omega - x_0 - 2$ by maximizing the contract value following a given strategy $[k_{\omega - x_0 - 2}, \infty) \times [k_{\omega - x_0 - 1}, \infty)$ for $k_{\omega - x_0 - 2}$, etc. Thus, by repeatedly maximizing the contract value for the optimal surrender boundaries, we are able to determine optimal stratefies.

5 Results

We use the numerical methods presented in Section 4 to calculate the risk-neutral value of variable annuities including GMWB or GLWB guarantees for a given guarantee fee φ. We call a contract and the corresponding guarantee fee *fair* if the contract's risk-neutral value equals the single premium paid, i.e. if $P = V_0 = V_0(\varphi)$.

Unless stated otherwise, we use a risk-free rate of interest *r* of 4%, a volatility σ of 15%, and a single premium *P* = 10,000. Furthermore, we let the age of the insured x_0 = 65, and the surrender fee *s* =10%. Moreover, we use best estimate mortality tables of the German society of actuaries (DAV 2004 R) for a male insured.

5.1 Determination of the Fair Guarantee Fee

To illustrate how the fair guarantee fee can be derived within our framework, in a first step, we analyze the influence of the guarantee fee on the value of contracts for three different kinds of GLWB guarantees. Contract 1 contains a plain vanilla guarantee, some guaranteed annual withdrawal amount of 5% of the single premium, contract 2 contains a roll-up at a rate of $i = 6\%$ for a maximum of 5 years, whereas an annual step-up is considered in contract 3. We assume deterministic customer behavior: For contract 1 and contract 3, the insured withdraws the guaranteed annual amount of 500 - beginning immediately in the first year of the contract; for contract 2, we assume that the insured person does not access his guaranteed amount for the first five years and then starts to withdraw the guaranteed amount of 669 per annum (which has been increased due to the roll-up feature). For all three versions, we assume that the contract is not surrendered.

Figure 1 shows the corresponding contract values as a function of the annual guarantee fee.

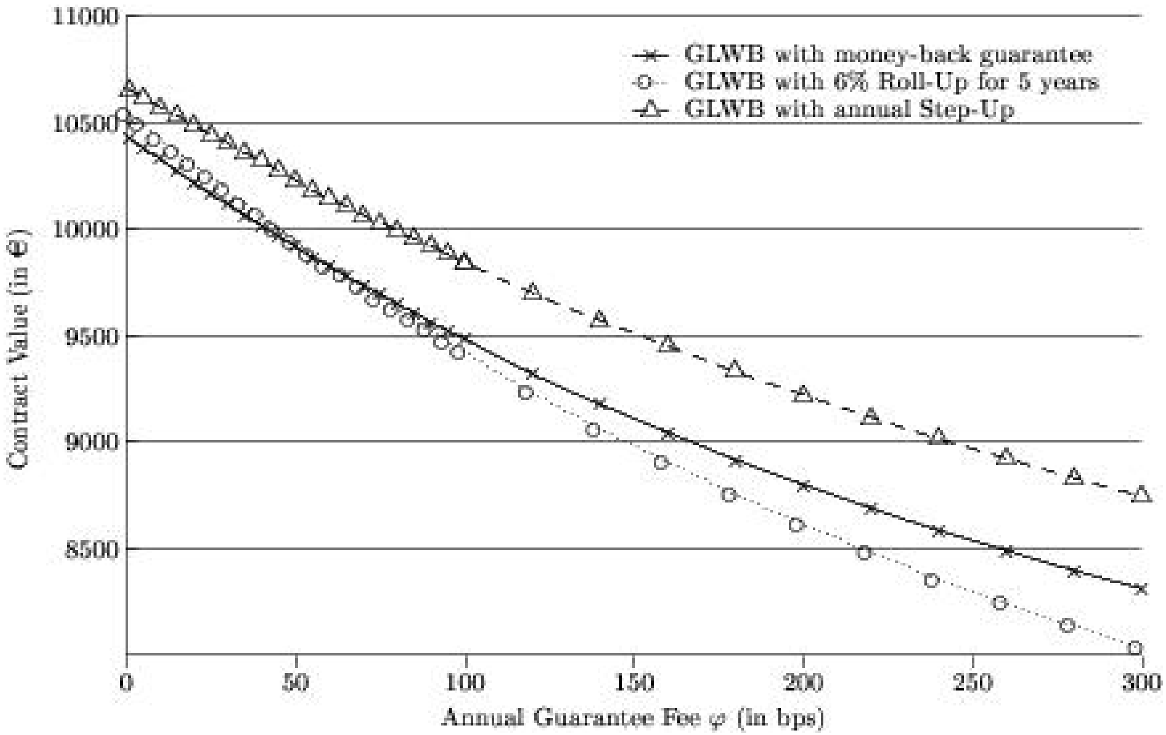

Figure 1 Contract values as a function of the guarantee fee for different versions of GLWB

For contract 1, the contract value equals 10,000 at a guarantee fee of 43, thus this is the fair guarantee fee. For contracts 2 and 3, the fair guarantee fee amounts to 48 bps and 80 bps, respectively.

5.2 Fair Guarantee Fees for Different Contracts

In this section we analyze contracts with different GLWB-versions. We start with contracts containing a roll-up feature and then move on to contracts with different step-up / ratchet features.

5.2.1 GLWB with Roll-Up

In this section, we analyze and compare three different contracts. Contract 1 guarantees lifelong withdrawals of 5% of the initial premium (no roll-up). Contract 2 provides an annual increase of the guaranteed withdrawal amount of 6% for a maximal period of 5 years, as long as no withdrawals are made. The third contract comes with a 5% roll-up rate and a roll-up period of 10 years. We consider all three contracts with and without a money-back GMDB option.

We assume the following policyholder behavior: In contract 1 the policyholder withdraws 5% of the initial premium up to his death starting immediately. For contract 2 and 3 the insured starts the withdrawals of 669 and 814, respectively at the end of the roll-up period (5 and 10 years, respectively) and then withdraws annually and lifelong the corresponding guaranteed annual withdrawal amount.

Table 1 shows the fair guarantee fee for these three contracts with and without an additional GMDB option.

contract	no roll-up		6% roll-up benefit for 5 years		5% roll-up benefit for 10 years	
strategy	**w/o DB**	**with DB**	**w/o DB**	**with DB**	**w/o DB**	**with DB**
Withdrawals of guaranteed amount starting at the end of roll-up period	**43 bps**	**48 bps**	**47 bps**	**52 bps**	**28 bps**	**35 bps**

Table 1: Fair guarantee fee for GLWB contracts with roll-up

The results show that it is not always optimal to postpone withdrawals until the end of the roll-up period even if guaranteed annual withdrawal amounts are increased in this case. The value of contract 3 under the assumed strategy is significantly lower than the value of contract 1. On the one hand, a roll-up rate exceeding the risk free rate leads to an increase in the value of the guarantee if withdrawals are postponed. On the other hand however, the insured becomes older during the waiting period. With increasing age, the expected number of withdrawals until death decreases and so does the value of the guarantee. Apparently, for our third contract, the latter effect is dominant.

The additional fee for death benefit (difference between columns “with DB” and “w/o DB”) increases as we move from contract 1 to 2 and 3. This is due to the fact that every withdrawal leads to a reduction of the GMDB value. Since we assume fewer and later withdrawals in contracts 2 and 3, the corresponding GMDB option is more valuable.

5.2.2 GLWB with ratchet guarantee

For the analysis of contracts with step-up feature, we again compare three different GLWB contracts. Contract 1 has no step-up feature and thus coincides with contract 1 from the previous subsection. In contract 2 a step-up is possible every 5th anniversary of the policy following the rules described in section 3. Finally, contract 3 comes with an annual step-up. Each of the contracts is analyzed with and without an additional money-back GMDB.

We assume the policyholder to annually withdraw the guaranteed amount until death (no surrender). If the guaranteed amount is increased by a step-up, we assume that the policyholder immediately increases the annual withdrawal amount to the new guaranteed amount.

The fair guarantee fees of the contracts are shown in Table 2.

contract / strategy	no step-up		5 year step-up		annual step-up	
	w/o DB	**with DB**	**w/o DB**	**with DB**	**w/o DB**	**with DB**
Withdrawals of guaranteed amount beginning in year 1	43 bps	48 bps	55 bps	62 bps	80 bps	88 bps

Table 2: Fair guarantee fee for GLWB contracts with different step-up features

Obviously the step-up feature provides additional value for the policyholders. Our analysis shows that the more frequently step-ups are provided, the higher the value of the feature and thus the fair guarantee fees are.

The additional fee for the death benefit (difference between columns “with DB” and “w/o DB”) is roughly equal for all three contracts (about 10% of the fair fee without GMDB). The effect observed in section 5.2.1 for contracts with roll-up can not be found here, since in scenarios where the GMDB option is in the money (declining fund values), the three contracts develop similarly.

5.3 Sensitivity Analysis with respect to the Guaranteed Annual Withdrawal Amount

In this section, we look at the influence of the annual maximum guaranteed withdrawal amount on the fair guarantee fee for the plain vanilla GLWB contract (i.e. neither a roll-up nor a step-up is included). We consider annual withdrawal amounts of x_{WL}=3%, x_{WL}=4%, x_{WL}=5%, and x_{WL}=6%. The fair guarantee fees are displayed in Table 3.

contract / strategy	x_{WL} = 3%	x_{WL} = 4%	x_{WL} = 5%	x_{WL} = 6%
Withdrawals of guaranteed amount annually beginning in year 1	3 bps	12 bps	43 bps	117 bps

Table 3: Influence of the maximum annual withdrawal amount on the fair guarantee fee for a GLWB contract

The maximum annual withdrawal amount notably influences the fair guarantee fee. Rather low annual withdrawal rates of 3% lead to a low fair guarantee fee of 3 bps, while a fee of 117 bps is necessary to back a GLWB option with 6% annual withdrawals. Unlike in a GMWB contract, the total withdrawal amount is not restricted. As a consequence, the influence of the annual maximum withdrawal amount in a GLWB contract is considerably higher than in a GMWB contract. For more details on the comparison of GMWB and GLWB see Section 5.7.

5.4 Sensitivity Analysis with respect to the Insured's Age

In the previous sections we assumed the insured to be 65 years old. We now determine the influence of the age on the contract value. In particular with respect to lifelong guarantees, the age is of significant influence on the contract value, since mortality rates roughly increase exponentially in age. We calculate contract values for insured persons aged 55, 65, 75, 85 and 95 years, respectively. The policyholder's strategy is assumed to be the same in all contracts, namely lifelong withdrawals of the guaranteed annual amount from the beginning of the contract and no surrender.

Table 4 shows the fair guarantee fees for these contracts.

Age strategy	55	65	75	85	95
Withdrawals of guaranteed amount annually (500) beginning in year 1	105 bps	43 bps	11 bps	~0.5 bps	~0.05 bps

Table 4: Fair guarantee fee for contracts with GLWB under different ages of the insurant

Since withdrawals are guaranteed lifelong, the fair guarantee fee is decreasing in the insured's age within the considered age interval. While the fair guarantee fee for a 55 year old amounts to 105 bps it decreases to less than 1 bp for an 85 or 95 year old person. These results explain why most of the current GLWB products require a minimum age of 60 years when withdrawals start. Some alternative to requiring for a minimum age would be to link the guaranteed annual withdrawal amount to the age at with withdrawals are actually started.

Thus, we now fix a guarantee fee of 50 bps and determine the fair withdrawal rate for different ages, i.e. the withdrawal rate for which the contract's risk-neutral value coincides with the premium paid. The fair withdrawal rates are displayed in Table 5.

age strategy	55	65	75	85	95
Withdrawals of guaranteed amount annually beginning in year 1	4.4%	5.2%	6.6%	9.2%	13.4%

Table 5: Fair withdrawal rate for GLWB contracts for a guarantee fee of 50 bps and different ages

Obviously, for the same fee, higher withdrawal rates can be guaranteed to older people. Whilst for a 55 year old only an annual withdrawal amount of 440 can be guaranteed, this amount increases to 1340 for a 95 year old.

5.5 Analysis of the Longevity Risk

As mentioned above, mortality rates are an important factor for the calculation of such contracts. So far, we used the mortality table DAV 2004R for our calculations. An increase of longevity that exceeds the trend embedded in this table would have a negative impact on the insurer's profitability. To analyze this risk we calculate the fair guarantee fee assuming different mortality probabilities. Table 6 gives the fair guarantee fee for the plain vanilla GLWB contract for different ages assuming the mortality table DAV 2004R as above and additionally under the assumption that mortality rates drop to 70% of the rates given in this table.

age / mortality table	55	65	75	85	95
DAV 2004R	105 bps	43 bps	11 bps	~0.5 bps	~0.05 bps
70% of DAV 2004R	138 bps	58 bps	20 bps	~1.5 bps	~0.05 bps

Table 6: Fair guarantee fee of a GLWB contract for different ages and different mortality scenarios

The results show that the value of lifelong withdrawal guarantees is rather sensitive with respect to longevity. For a 55-year old, the fair guarantee fee increases by almost one third to 138 bps, and for an 75-year old, the fair guarantee fee doubles if mortality rates drop by 30%.

5.6 Sensitivity Analysis with respect to Capital Market Parameters

In this section we analyze the influence of the capital market parameters r and σ on the contract value. Up to now we based our calculations on the assumption of r=4% and σ=15%. We consider the plain vanilla GLWB contract assuming a customer who withdraws the guaranteed annual amount and does not surrender. We vary the risk-free rate of interest r as well as the volatility σ.

Table 7 shows the fair guarantee fee for different combinations of the capital market parameter values.

risk-free rate / volatility	r=3%	r =4%	r =5%
σ = 10%	48 bps	19 bps	7 bps
σ = 15%	85 bps	43 bps	20 bps
σ = 20%	124 bps	69 bps	41 bps

Table 7: Influence of the capital market parameters *r* and σ on the fair guarantee fee for a GLWB contract

As expected, the fair guarantee fee is decreasing in the risk-free rate of interest and increasing in the volatility since, on the one hand, the risk-neutral value of a guarantee decreases with increasing interest rates; and, on the other hand, options are more expensive when volatility increases. Changes in volatility have a tremendous impact on the option values and, thus, on the fair guarantee fee.

At inception of the contract and with some products also during the term of the contract, the insured has the possibility to influence the volatility by choosing the underlying fund from a predefined selection of mutual funds. Since for some products offered in the market the fees do not depend on the fund choice, this possibility presents another valuable option for the policyholder. For any risk-free rate r, the fair guarantee fee for σ = 20% is more than twice as high as the fee for σ =10%. Thus, one important risk management tool for insurers offering variable annuity guarantees is the strict limitation and control of the types of underlying funds offered within these products.

An alternative would be to link the guaranteed annual withdrawal amount to the fund's expected volatility. One insurer gives a guaranteed annual withdrawal amount of 5% of the single premium paid for three different funds with different stock ratios between. However, the fee increases in the fund's stock ratio. Besides, a fourth fund with an even higher stock ratio is offered. However, the guaranteed annual withdrawal amount is reduced to 4.5% if this fund is chosen.

Therefore, in a next step we determine the fair withdrawal rate for different capital market parameters for a guarantee fee of 50 bps. The results are shown in Table 8.

risk-free rate volatility	r=3%	r =4%	r =5%
σ = 10%	4.9%	5.7%	6.6%
σ = 15%	4.6%	5.2%	5.8%
σ = 20%	4.1%	4.7%	5.1%

Table 8: Influence of the capital market parameters r and σ on the fair withdrawal rate for a GLWB contract (guarantee fee 50 bps)

Obviously, for a given guarantee fee the fair withdrawal rate is increasing in the risk-free rate of interest and decreasing in the volatility.

5.7 Comparison of GLWB and GMWB

As mentioned above, GLWB-options are a recent variation of GMWB-products. In what follows, we compare the two product types.

In a first step, we analyze the value of a contract with GLWB under different withdrawal rates. The values for contracts under the withdrawal rates 3%, 4% and 5% as a function of the annual guarantee fee are shown in Figure 2.

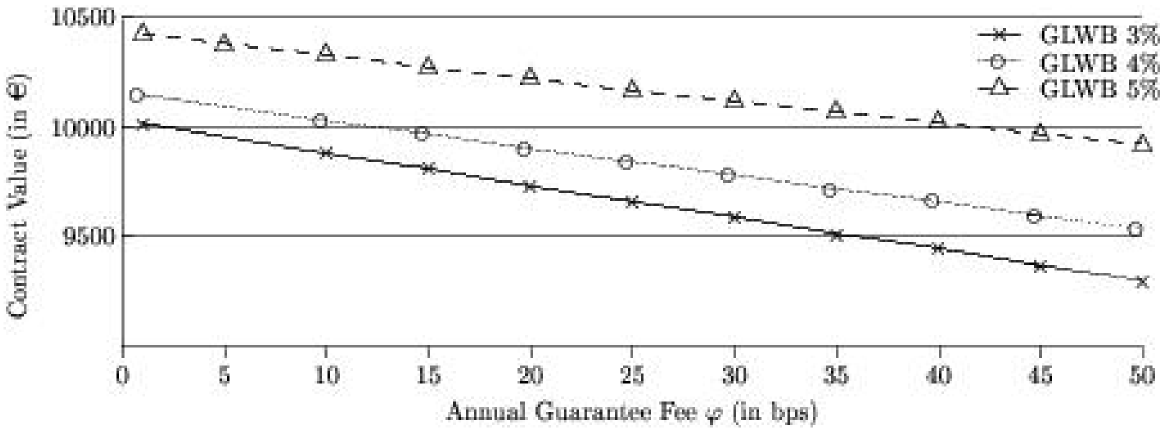

Figure 2 Value of GLWB contracts as a function of the annual guarantee fee

The three curves are nearly parallel. The results show, as expected, that the contract is significantly more valuable for a higher annual withdrawal rate.

In a second step, we analyze the influence of the annual withdrawal rate on a GMWB contract. Figure 3 displays the contract values for withdrawal rates of 5%, 6% and 7%.

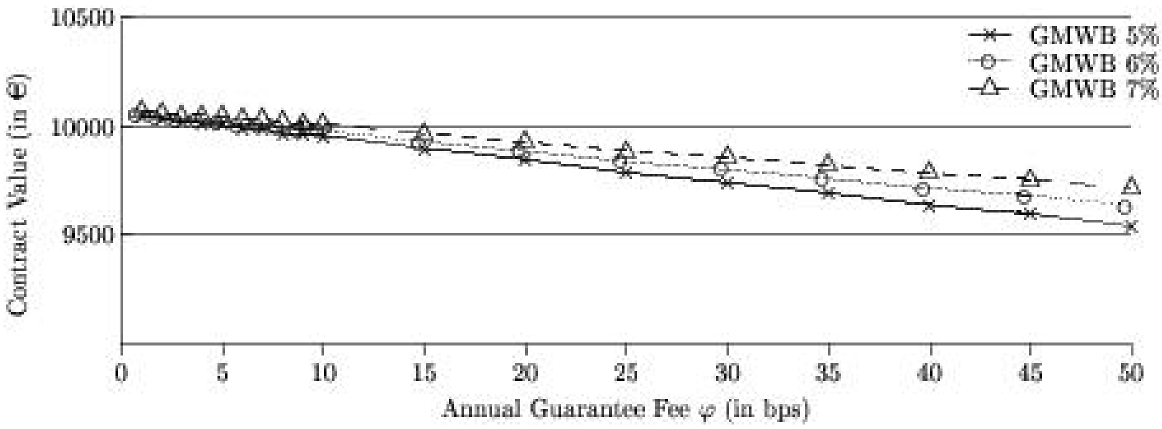

Figure 3 Value of GMWB contracts as a function of the annual guarantee fee

For small guarantee fees, the contract values are very close; only for higher guarantee fees, the gap between the different contracts increases and the contract with a withdrawal rate of 7% is significantly more valuable than the ones with lower rates.

The obvious difference between Figure 2 and Figure 3 is due to the different structure of the GMWB and GLWB options. In a GMWB the total withdrawal amount is limited (for example by the initial premium); a higher annual withdrawal rate in a contract with GMWB results in a lower number of withdrawals. By contrast, high withdrawals that are guaranteed within a GLWB are guaranteed until death and therefore increase the value of the option significantly.

Finally, we compare the fair withdrawal rate within the GLWB and the GMWB option for given guarantee fees. As a policyholder's strategy we assume that the annual guaranteed amount is withdrawn until death within a GLWB and until the total withdrawal amount has been reached within a GMWB. For the analysis we choose a guarantee fee of 12 bps, which is the fair guarantee fee for a common product in the US-market, a 7% GMWB; further, we examine the fair guarantee fee for the GLWB with a withdrawal rate of 5%, which is 43 bps.

The results are shown in Table 9.

Benefit Guarantee fee	Guaranteed Lifetime Withdrawal Benefit	Guaranteed Minimum Withdrawal Benefit
12 bps	4.0%	7.0%
43 bps	5.0%	12.5%

Table 9: Fair withdrawal rate for GLWB and GMWB under different guarantee fees

For the guarantee fee of 12 bps, the fair withdrawal rate of a GLWB-option is 4%. This means, for the same guarantee fee the policyholder can withdraw 400 annually within a GLWB and 700 in a GMWB. Furthermore, a contract with 5% GLWB corresponds to a GMWB contract with 12.5% guaranteed annual withdrawal.

This valuation is of relevance, as some insurers offer their clients the choice between a GMWB and a GLWB with different withdrawal rates.

5.8 Results under Optimal Customer Behavior

In this last subsection, we show the results for optimal customer behavior that have been derived using the methods described in Section 4. First we calculate the bounds k_t for the optimal strategy in a contract that contains a GLWB for a guarantee fee of 50 bps.

Figure 4 displays the bounds k_t, which start at about 18,700 € in the first year and decrease slightly in the beginning and stronger as the insured's age increases.

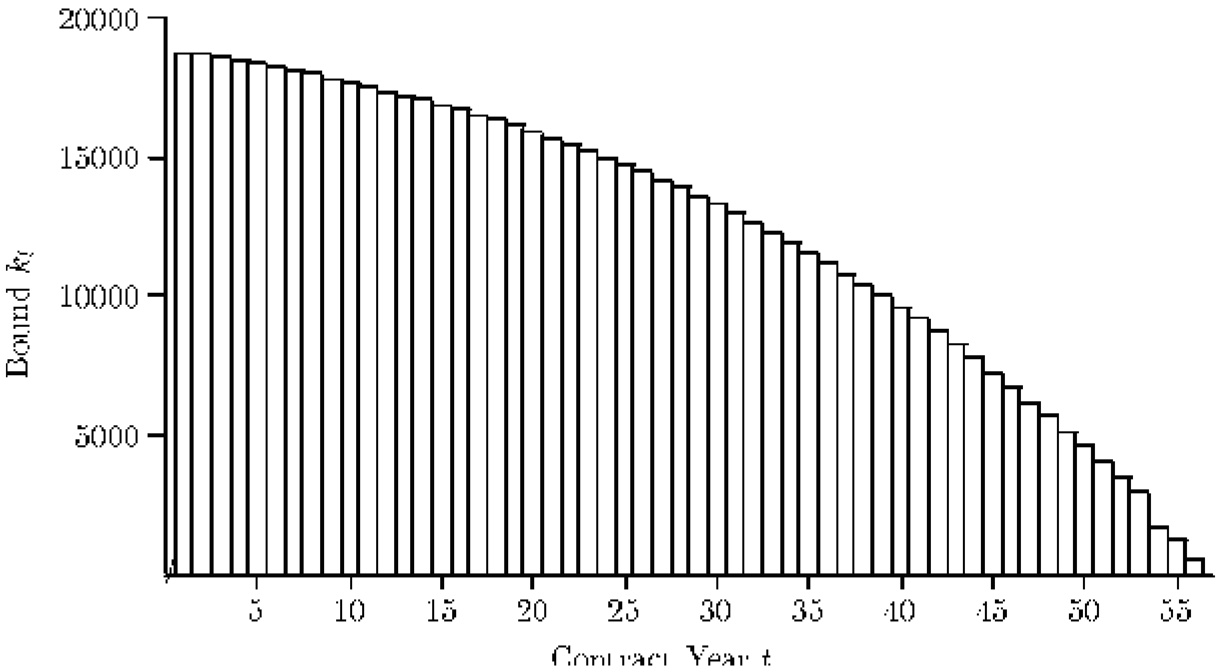

Figure 4 Optimal Surrender Strategy for a GLWB contract and a guarantee of 50 bps

The fair guarantee fee under optimal policyholder behavior amounts to 49 bps. This is slightly lower than the current fees of most insurance companies for this benefit. It is notable that the fair guarantee fee under optimal customer behavior differs only slightly from the fair fee assuming deterministic behavior (annual withdrawal, no surrender). Bauer et al (2006) found much larger differences between deterministic and optimal client behavior for GMAB, GMIB and GMWB products. The reasons for this is, as we have seen above, that in GLWB products, the client has fewer choices than in other products since optimal behavior is always characterized by either withdrawing the guaranteed amount or surrendering the contract.

6 Summary and Outlook

Guaranteed lifelong withdrawal benefits are the latest innovation in the variable annuity market. The products are very popular since they cover longevity risk like a regular annuity but combine this feature with a permanent availability of the remaining account value (if positive).

However, our analyses have shown that such products are rather risky for the insurer: Whilst these products are much less sensitive with respect to client behavior than other guarantees typically embedded in variable annuities (cf. Bauer et al. (2006)), the sensitivity with respect to changes in interest rates and fund volatilities as well as mortality rates is significant. This is particularly dangerous considering the long time horizon of the products and the fact that according to market survey (cf. Lehman Brothers (2005)) often only delta risk is hedged.

Since our asset model is rather simple, a worthwhile extension might be an analysis of such products in a Lévy-type framework with stochastic interest rates. Also, in particular for GLWB products, it would be interesting to see how our results change in a model with stochastic mortality rates (cf. e.g. Cairns et al. (2005) or Bauer et al. (2007)).

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