Life-cycle Funds: Much Ado about Nothing?

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Abstract

The core idea of life-cycle funds or target-date funds is to decrease the fund's equity exposure and conversely increase its bond exposure towards the fund's target date. Such funds have been gaining significant market share and were recently set as default choice of asset allocation in numerous defined contribution schemes or related old age provision products in several countries. Hence, an assessment of life-cycle funds' risk-return profiles – i.e. the probability distribution of returns – is essential for sustainable financial planning of a large group of investors. This paper studies the risk-return profile of life-cycle funds in particular compared to simple balanced or lifestyle funds that apply a constant equity portion throughout the fund's term instead.

In a Black-Scholes model, we derive balanced funds that reproduce the risk-return profile of an arbitrary life-cycle fund for single and regular contributions. We then analyze the accuracy of our results under more complex asset models with stochastic interest rates, stochastic equity volatility and jumps. We further show that frequently used "rule of thumb approximations" that only take into account the life-cycle fund's average equity portion are not suitable to approximate a life-cycle fund's risk-return profile. Our results on the one hand facilitate sustainable financial planning and on the other hand challenge the very existence of life-cycle funds since appropriately calibrated balanced funds can offer a similar (often dominating) risk-return profile.

Keywords: Life-cycle Funds, Balanced Funds, Stochastic Modeling, Risk-Return Profiles.

1 Introduction

The demographic transition resulting from a continuing increase in life expectancy combined with rather low fertility rates constitutes a severe challenge for government-run pay-as-you-go pension systems in many countries. Therefore, the importance of funded private and/or occupational old age provision has been increasing and is likely to continue to increase. In particular, occupational pension plans provide an essential building block for retirement income. Its importance is likely to further increase due to mandatory "auto-enrollment" in these plans in many countries.[1] Hence, the underlying pension plan's performance will have a significant impact on retirement income. A large number of pension plans allocate their contributions to so-called life-cycle or target-date funds[2], since – according to Charlson et al. (2010) – 96% of the large[3] pension plans in the US selected life-cycle funds as their default asset allocation and most of the money allocated in these funds is left in the same fund until retirement. Further, Charlson et al. (2010) state that *"[…] by most measures, target-date funds have been a smashing success".* Also in the retail sector, life-cycle funds gain popularity as e.g. pointed out in Viceira (2008) who finds that assets under management of life-cycle funds in the US have increased from \$1bn in 1996 to \$120bn in 2006. Therefore, an appropriate assessment of life-cycle funds' risk-return profiles – i.e. the potential losses or gains – is highly relevant for sustainable financial planning of a large group of institutional as well as retail investors. For assessing the risk-return profile of old age provision products, Graf et al. (2012) propose a general methodology and further derive quantitative results for some product types offered in many markets. Among others, they compare an artificial balanced fund[4] and an artificial life-cycle fund and particularly focus on the effect of different premium payment patterns.

In line with the growing success in terms of assets under management, academia has recently started to investigate life-cycle funds in more detail. Generally, two main streams of research

[1] The process of automatically enrolling employees to some pension plan has e.g. been implemented by the United States or the United Kingdom.

[2] A life-cycle fund invests in risky and riskless assets according to a pre-specified "glide path" specifying how the asset allocation deterministically changes as the target date approaches.

[3] Defined as including more than 5,000 employees.

[4] A balanced fund invests in a constant mix of equity and bonds.

can be identified. The first one is concerned with comparing possible returns of artificial life-cycle funds and artificial balanced funds. As a starting point, Blake et al. (2001) provide a Value-at-Risk based analysis of different investment strategies including a balanced and a life-cycle strategy within a defined contribution plan setting. In addition, in the context of mutual funds, various authors such as Schleef and Eisinger (2007), Spitzer and Singh (2008), Pang and Warshawsky (2008) or Pfau (2010) – to name only a few – analyze the results of some life-cycle strategies compared to some balanced funds using different methodologies such as expected utility or shortfall measures in the pre- and post- retirement phase. Due to the rather artificial choice of the balanced and life-cycle funds under investigation in the different papers, their conclusions differ from preferring life-cycle funds over balanced funds (e.g. Pfau, 2010) or vice versa (e.g. Spitzer and Singh, 2008 or Schleef and Eisinger, 2007). In line with the controversy in results and conclusions above, e.g. Pang and Warshawsky (2008) are indifferent between preferring life-cycle or balanced funds from an investor's point of view.

The second strain of literature is concerned with finding "optimal" life-cycle funds. E.g. Cairns et al. (2006) derive optimal path-dependent life-cycle strategies applying stochastic control techniques to maximize expected utility in a defined contribution plan environment. In line with their results, Basu et al. (2009) or Antolín et al. (2010) conclude that path-dependent strategies are superior to deterministic life-cycle funds using simulation techniques and measuring shortfall. In contrast to determining the optimal path-dependent strategy, e.g. Maurer et al. (2007) and Gomes et al. (2008) derive the optimal deterministic life-cycle strategy, i.e. the optimal pre-specified glide path by maximizing expected utility and conclude that the "classical" glide path of starting with rather high equity exposure and then reducing the equity exposure over time is optimal. However, Basu and Drew (2009) prefer a "contrarian" life-cycle strategy – i.e. starting with moderate equity exposure and increasing the equity exposure towards the target date – over the conventional life-cycle due to their so-called "portfolio size effect".[5]

Summarizing, a sound opinion or common understanding, whether a balanced fund or a life-cycle fund with some *deterministic* glide path is preferable does not exist. However, Poterba

[5] The portfolio size effect as introduced by Basu and Drew (2009) means the general increase of invested volume when the contract's maturity approaches and regular contributions are considered.

et al. (2009) argue in their quantitative comparison that *"the results suggest that the distribution of retirement wealth associated with typical life-cycle investment strategies is similar to that from age-invariant asset allocation strategies that set the equity share of the portfolio equal to the average equity share in the life-cycle strategies",* a "rule of thumb" that has already been proposed by Lewis (2008).

The scope of this paper is therefore to investigate the (presumed) difference of balanced and life-cycle funds in more detail. However, in contrast to the research so far we do not compare artificial balanced and life-cycle strategies. Instead, for any pre-specified life-cycle fund we derive balanced funds that closely match the life-cycle fund's risk-return profile, i.e. the probability distribution of returns at the end of the investment horizon. This gives a better understanding of the "difference" between life-cycle and balanced funds and provides investors with an easy risk assessment of any given life-cycle investments: Its risk coincides with the risk of the matching balanced fund. Moreover, our results challenge the very existence of life-cycle funds, since we show that for any given life-cycle an (approximately) replicating or even dominating balanced funds does exist.

We start with deriving closed form solutions for a matching or dominating balanced fund in a Black-Scholes economy for single and regular contributions to the funds and further critically investigate the rule of thumb proposed by Lewis (2008). Then, we challenge the accuracy of our approximations under more complex asset models with stochastic interest rates, stochastic equity volatility and jumps.

The remainder of this paper is organized as follows: Section 2 describes our modeling approach of life-cycle and balanced funds whereas Section 3 introduces the financial models considered. Section 4 and Section 5 present our results for single and regular contributions respectively. Finally, Section 6 concludes.

2 Modeling life-cycle and balanced funds

Life-cycle and balanced funds generally invest in a mix of equity and bonds. A balanced fund has a constant portion $x_S \in [0,1]$ of its capital invested in equity and the remaining part in bonds. We further assume a fixed duration d for the bond portion and therefore model the bond portfolio by a zero-coupon bond investment with time to maturity d. In contrast to the balanced fund, a life-cycle fund applies a time-dependent (however not path-dependent) asset allocation strategy, where $\left(x_{S,t}\right)_t \in [0,1]$ denotes the equity portion at time t. We assume a continuous rebalancing between equity and bonds and further deduct a fund management fee on a continuous basis.

Besides deriving some theoretical results, we will consider three different life-cycle strategies in additional numerical analyses. We assume their asset mix to be constant throughout each year. For given time to maturity $T > 1$, the three considered strategies are:

- Strategy A defines a "classical" life-cycle strategy that starts with a complete investment in equity, i.e. $x_{S,0} = 1$ and each year linearly decreases its equity exposure until it arrives at a complete investment in the zero-bond in the last year, i.e. $x_{S,T-1} = 0$.

- In contrast, strategy B reverses the life-cycle and is therefore often referred to as "contrarian" strategy (cf. Basu and Drew, 2009): It starts with $x_{S,0} = 0$ and each year linearly increases its exposure to the risky asset finally arriving at $x_{S,T-1} = 1$.

- Finally, for investigating the accurateness of the approximations derived in the theoretical part of our analyses, strategy C applies a very artificial and extreme non-standard "life-cycle" model where the life-cycle fund alternates between a complete investment in equities and bonds on a yearly basis, i.e. we set $x_{S,0} = 1, x_{S,1} = 0, x_{S,2} = 1, \ldots$.

By comparing expected outcome and various percentiles of the resulting wealth distribution, Basu and Drew (2009) argue that contrarian strategies are superior to the classical life-cycle strategy due to the "portfolio size effect". In Sections 4 and 5 we will show that contrarian strategies in general offer higher returns but also significantly increase the risk especially when regular contributions are considered. This is an indication that contrarian strategies are not superior but rather suitable for less risk averse clients. However, the scope of this paper is not on comparing the performance of different life-cycle strategies but rather on matching them with appropriately calibrated balanced funds.

3 Financial models

In a first step, we apply a version of the analytically tractable Black-Scholes model (cf. Black and Scholes, 1973) which will be denoted by "BS" in what follows. Second, we consider a more complex model denoted by "CIR-SV", additionally allowing for stochastic interest rates using a Cox-Ingersoll-Ross model (cf. Cox et al., 1985) and modeling stochastic equity volatility using a version of the Heston-model (cf. Heston, 1993). Finally, we add jumps to the equity process by means of the Merton-model (cf. Merton, 1976). This hybrid stochastic volatility jump diffusion model is then denoted by "CIR-SVJD".

Let $r(t)$ denote the short-rate and $S(t)$ denote the equity's spot price at time t. The (BS) model is then entirely described by defining the *real-world*[6] dynamics of the underlying equity process as

$$dS(t) = S(t)\big((r(t) + \lambda_S)dt + \sigma_S dW_1(t)\big) \text{ and } r(t) = r, \forall t$$

where λ_S denotes the equity risk premium, $\sigma_S > 0$ is the annualized volatility of the equity process and r gives the non-random constant risk-less[7] interest rate. Further, $W_1(t)$ is a Brownian Motion under the considered probability measure.

The (CIR-SV) model adds stochastic interest rates and stochastic equity volatility to the (BS) model and is summarized by

$$\begin{aligned} dS(t) &= S(t)\left((r(t) + \lambda_S)dt + \sqrt{V(t)}dW_1(t)\right) \\ dV(t) &= \kappa_V\big(\theta_V - V(t)\big)dt + \sigma_V\sqrt{V(t)}dW_2(t) \\ dr(t) &= \kappa_r\big(\theta_r - r(t)\big)dt + \sigma_r\sqrt{r(t)}dW_3(t) \end{aligned}$$

where $W_2(t)$ and $W_3(t)$ are Brownian Motions under the probability measure. We further assume $dW_1(t)dW_3(t) = dW_2(t)dW_3(t) = 0$ and $dW_1(t)dW_2(t) = \rho dt$ with $\rho \in [-1,1]$ driving the correlation between the spot price and its instantaneous variance $V(t)$. Although the random innovations of interest rate and equity markets are assumed to be uncorrelated (i.e. $dW_1(t)dW_3(t) = 0$), interest rate and equity markets are implicitly correlated due to the applied spread model.

The (CIR-SVJD) model finally extends the (CIR-SV) model by allowing for jumps in the equity process and is given by

$$\begin{aligned} dS(t) &= S(t)\left((r(t) + \lambda_S)dt + \sqrt{V(t)}dW_1(t) + dJ(t)\right) \\ dV(t) &= \kappa_V\big(\theta_V - V(t)\big)dt + \sigma_V\sqrt{V(t)}dW_2(t) \end{aligned}$$

[6] Risk-return profiles should always be assessed under the objective probability measure (cf. Graf et al. 2012).

[7] We do not consider default risk in this paper.

$$dr(t) = \kappa_r(\theta_r - r(t))dt + \sigma_r\sqrt{r(t)}dW_3(t)$$

where $dJ(t) = Z(t)dN(t)$ gives the jump's dynamics. Within this setting $N(t)$ is a Poisson counter with intensity λ indicating the occurrence of a jump and $Z(t)$ gives the random jump size. In the spirit of Merton (1976), we assume that the jump sizes are normally distributed and hence set $Z(t) \sim \mathcal{N}(\mu_Z, \sigma_Z^2)$.

In the models with stochastic interest rates, zero-bond prices $P(t,d)$ at time t with time-to-maturity d can be derived using standard no-arbitrage arguments[8]: $P(t,d) = A(d)e^{-B(d)r(t)}$ with $A(d) = \left[\frac{2\cdot h\cdot\exp\left((\tilde{\kappa}_r+h)\cdot {}^{d}/_{2}\right)}{(\tilde{\kappa}_r+h)\cdot(\exp(h\cdot d)-1)+2\cdot h}\right]^{\frac{2\tilde{\kappa}_r\tilde{\theta}_r}{\sigma_r^2}}$ and $B(d) = \frac{2\cdot(\exp(h\cdot d)-1)}{(\tilde{\kappa}_r+h)\cdot(\exp(h\cdot d)-1)+2\cdot h}$ where $h = \sqrt{\tilde{\kappa}_r^2 + 2\cdot\sigma_r^2}$, $\tilde{\kappa}_r = \kappa_r + \lambda_r\sigma_r$, $\tilde{\theta}_r = \frac{\kappa_r\theta_r}{\kappa_r+\lambda_r\sigma_r}$ and λ_r denotes the market price of interest rate risk.

In the numerical analyses in Sections 4 and 5, we adopt capital market parameters from Graf et al. (2012) summarized in Table 1.

κ_r	θ_r	σ_r	λ_r	κ_V	θ_V	σ_V	ρ	λ_S
20%	4.5%	7.5%	0	475%	$(22\%)^2$	55%	-57%	3%

Table 1: Capital market parameters (without jump parameters)

For reasons of consistency with the (BS) model, we further assume the initial short rate and the initial spot volatility (or variance) being equal to their long-term expectations and consequently set $r(0) = 4.5\%$ and $V(0) = 22\%^2$.[9]

[8] Cf. e.g. Bingham and Kiesel (2004).

[9] Note, we could also assume some different short-rate at $t = 0$ which then implies a non-constant but time-dependent drift term in the (BS) model and thus slightly changes the theoretical results in Sections 4 and 5.

Finally, the jump parameters are taken from Eraker (2004) and summarized in Table 2.

λ	μ_Z	σ_Z
0.5	- 0.4%	6.6%

Table 2: Jump model parameters

The Black-Scholes model is then parameterized accordingly by setting $r = r(0)$ and $\sigma_S = \sqrt{V(0)}$. Further, zero-bond prices in the (BS) model are given by $P(t,d) = \exp(-r \cdot d)$.

4 Single contribution

First, we analyze a single contribution to an arbitrary life-cycle fund as described in Section 2. We derive some closed form solutions within the (BS) model in Section 4.1. Section 4.2 then analyzes if these solutions can serve as approximations under the more complex financial models introduced above.

4.1 Matching a balanced fund in the (BS) model

In what follows, let $V_{LF}(t)$ denote the spot price at time t of a life-cycle fund with time-dependent equity allocation $\left(x_{S,t}\right)_t$, target-duration d and management fee $c_{LF} \geq 0$. In a Black-Scholes framework, the dynamics of $V_{LF}(t)$ are then given by

$$dV_{LF}(t) = V_{LF}(t)\left(\left(x_{S,t}(r+\lambda_S) + (1-x_S)r - c_{LF}\right)dt + x_{S,t}\sigma_S dW_1(t)\right)$$

which can be solved to

$$V_{LF}(t) = \exp\left(\int_0^t \left(x_{S,u}(r+\lambda_S) + \left(1-x_{S,u}\right)r - c_{LF} - \frac{1}{2}\left(x_{S,u}\sigma_S\right)^2\right)du + \int_0^t \left(x_{S,u}\sigma_S\right)dW_1(u)\right)$$

Further, let $V_{BF}(t)$ denote the spot price at time t of a balanced fund with constant equity portion x_S, the same target-duration d and management fee c_{BF}. $V_{BF}(t)$ can similarly be written as

$$V_{BF}(t) = \exp\left(\left(x_S(r+\lambda_S) + (1-x_S)r - c_{BF} - \frac{1}{2}(x_S\sigma_S)^2\right)t + \int_0^t (x_S\sigma_S)dW_1(u)\right)$$

PROPOSITION 1

Consider an investment horizon of T years and an arbitrary life-cycle fund characterized by its glide-path $\left(x_{S,t}\right)_t$ and its management fee c_{LF}. If the risk premium λ_S is non-negative we obtain

a) a unique balanced fund characterized by its equity portion $x_S \in [0,1]$ and management fee $c_{BF} \geq 0$ which exactly replicates the life-cycle fund's probability distribution of returns at maturity T.
b) $c_{BF} \geq c_{LF}$.
c) a balanced fund defined by the equity portion x_S of the replicating balanced fund obtained in part a) and the management fee c_{LF} of the life-cycle fund stochastically dominates the life-cycle fund.

Proof:

Applying above calculations, both $V_{LF}(T)$ and $V_{BF}(T)$ follow a lognormal distribution and their distributions hence coincide if and only if $\mathbb{E}_{\mathcal{P}} V_{LF}(T) = \mathbb{E}_{\mathcal{P}} V_{BF}(T)$ and $\mathbb{Var}_{\mathcal{P}} V_{LF}(T) = \mathbb{Var}_{\mathcal{P}} V_{BF}(T)$. From standard algebra and standard results for log-normal distributions it follows that the considered funds' probability distributions of returns coincide if and only if

$$\int_0^T (x_S \sigma_S)^2 du \stackrel{!}{=} \int_0^T \left(x_{S,u}\sigma_S\right)^2 du \Rightarrow x_S = \sqrt{\frac{1}{T}\int_0^T x_{S,u}^2 du}$$

and

$$T(x_S(r+\lambda_S) + (1-x_S)r - c_{BF}) \stackrel{!}{=} \int_0^T \left(x_{S,u}(r+\lambda_S) + \left(1-x_{S,u}\right)r - c_{LF}\right) du$$

$$\Rightarrow c_{BF} = c_{LF} + \lambda_S \left(x_S - \frac{1}{T}\int_0^T x_{S,u} du\right).$$

From the equations above, the balanced fund's equity portion x_S and its management fee c_{BF} can be derived such that the distribution of the life-cycle and balanced fund *at the end of the investment horizon* coincide.

However, it is not yet clear if the calibrated equity portion and management fee actually define a "valid" balanced fund, i.e. a balanced fund with an equity portion between 0% and

100% and a non-negative management fee. First, we obtain $x_S \in [0,1]$ by construction (cf. above equation) and second, the Cauchy-Schwartz inequality yields $\int_0^T \frac{1}{T} x_{S,u} du \leq \sqrt{\int_0^T \frac{1}{T^2} du \int_0^T x_{S,u}^2 du} = \sqrt{\frac{1}{T}\int_0^T x_{S,u}^2 du} = x_S$. Hence, if the risk premium λ_S is non-negative (as it is expected to be), $c_{BF} \geq c_{LF} \geq 0$ immediately follows.

In addition, setting up a balanced fund with equity portion x_S and the same management fee as the life cycle fund ($c_{BF} \coloneqq c_{LF}$) yields a balanced fund with same volatility as the life-cycle fund and a higher return which in the setting of log-normal distributed random variables means stochastic dominance.

□

Consequences of Proposition 1

For single contributions – neglecting parameter and model risk – the very existence of life-cycle funds is challenged since for any life-cycle fund, a balanced fund exists that delivers exactly the same risk-return profile at maturity although it is more expensive than the life-cycle fund. Hence, if this balanced fund is equipped with the same fee as the life-cycle fund, it even dominates the life-cycle fund.

Since this dominating fund is uniquely defined by its equity portion which is independent of the parameters of the Black-Scholes model and the life-cycle fund's management fee, the construction of this balanced fund is not subject to parameter risk.

For the capital market parameter set as defined in Section 3, Table 3 gives equity portion and management fee of the balanced funds calibrated to the sample life-cycle strategies A, B and C as introduced in Section 2. In this example, we set the life-cycle funds' management fee to $c_{LF} = 1.3\%\ p.a.$ and assume an investment horizon of $T = 12$ years.

Strategy	Equity portion x_S	Management fee c_{BF}
A	59.04%	1.57% $p.a.$
B	59.04%	1.57% $p.a.$
C	70.71%	1.92% $p.a.$

Table 3: Balanced funds' calibration – single contribution

Note that in case of a single contribution and a Markovian capital market model – e.g. the considered (BS) model – strategies A and B yield the same risk-return profile and therefore obviously correspond to the same balanced fund. Further, strategy C results in the riskiest and most expensive balanced fund (in terms of equity portion and management fee, respectively). Further, it is worth noting that the average equity portion of the considered life-cycle strategies A, B and C over time equals 50%. Hence, the rule of thumb as proposed by Lewis (2008) would approximate all three funds with the same balanced fund although the differences are quite substantial (cf. Table 3 and Figure 1).

Figure 1 shows the probability densities of the internal rate of return of the three different life-cycle strategies (or equivalently the corresponding balanced funds as given in Table 3) assuming the (BS) model and an investment horizon of $T = 12$ years. We also display a balanced fund with a constant equity portion of 50% and a management fee of 1.3% $p.a.$ applying the rule of thumb calibration technique.[10]

[10] The rule of thumb calibration is obtained by setting $x_S = \frac{1}{T}\int_0^T x_S \, ds$ and $c_{BF} = c_{LF}$.

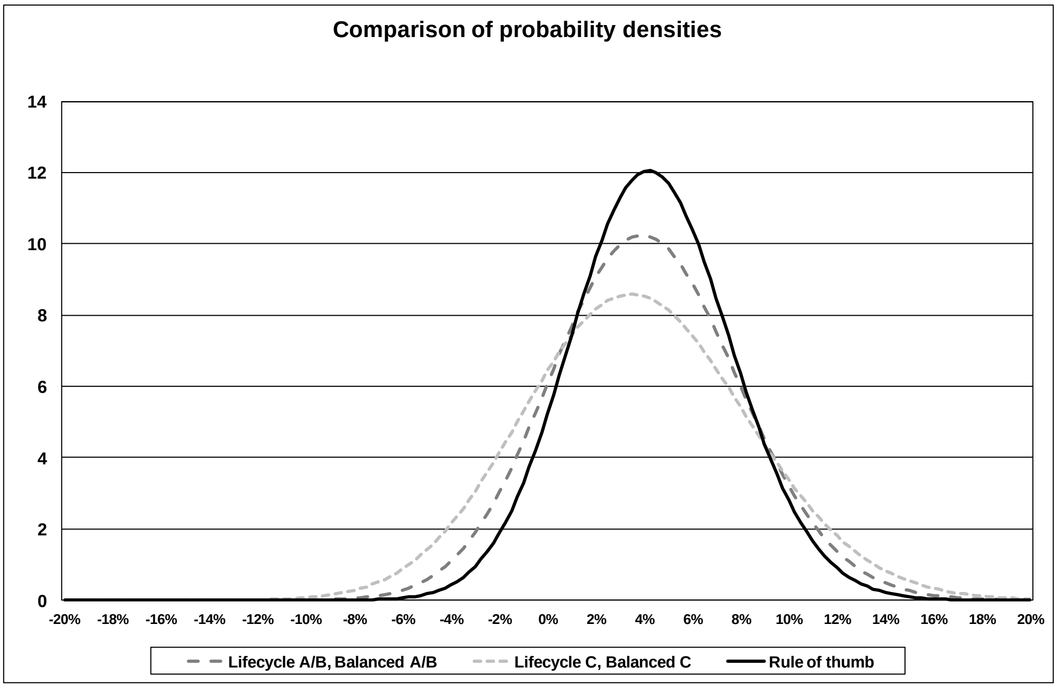

Figure 1: Life-cycle vs. balanced fund – single contribution (BS)

Hence, even in the "simple" single contribution case, the rule of thumb completely fails to approximate the risk-return profiles of the considered life-cycle funds and is therefore of only very limited explanatory value.

Finally, Table 4 summarizes some percentiles of the underlying probability distributions of returns, again stressing the weaknesses of the rule of thumb approximation.

Strategy	5%	25%	Median	75%	95%
Life-cycle / Balanced A/B	−2.29%	1.33%	3.92%	6.58%	10.53%
Life-cycle / Balanced C	−3.83%	0.45%	3.54%	6.73%	11.48%
Rule Of Thumb	-1.13%	1.96%	4.17%	6.43%	9.76%

Table 4: Key figures – single contribution (BS)

4.2 Model risk

We will now analyze the impact of model risk by applying the more complex financial models as introduced in Section 3. Since under these models no analytical solutions exist, we compare the life-cycle funds' and balanced funds' risk-return profiles by means of Monte-Carlo simulation techniques.[11] We perform a daily rebalancing of the assets and daily deduction of the management fee and assume that each month consists of 21 trading days. Further, for simulating the underlying stochastic processes $\left(S(t), V(t), r(t)\right)$ we assume a step size of $\Delta t = \frac{1}{12 \cdot 21 \cdot 5}$, i.e. we simulate these processes "5 times a trading day". Considering the projection of the underlying square root processes $\left(V(t), r(t)\right)$, we apply the "full truncation method" as described by Andersen (2008) and introduced respectively analyzed in great detail by Lord et al. (2010).

As an approximation for the life-cycle funds, we still use the balanced funds derived in the (BS) model (cf. Table 3) and now analyze whether these funds still constitute good approximations for the life-cycle funds when more randomness is introduced in the financial models.

Note that the (CIR-SV) model adds stochastic interest rates and stochastic volatility to the (BS) model. Figure 2 illustrates the empirical probability density of the considered life-cycle and balanced funds after applying a Kernel-smoother using 20,000 Monte-Carlo trajectories.

[11] Cf. e.g. Fishman (1996) or Glassermann (2004) for a thorough introduction of Monte-Carlo simulation techniques.

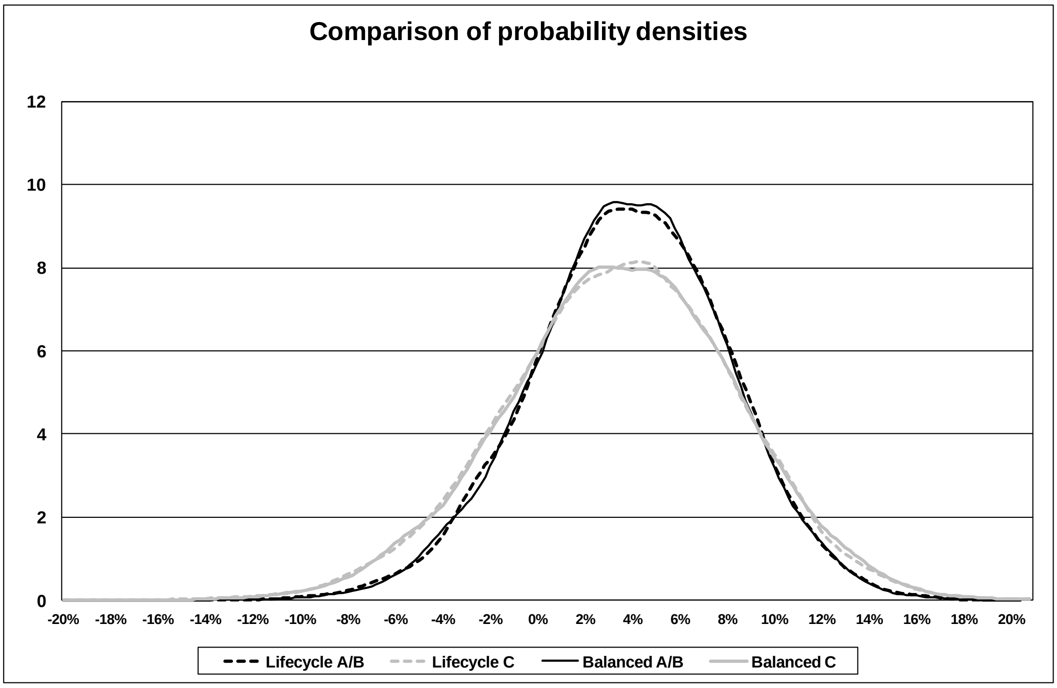

Figure 2: Life-cycle vs. balanced fund – single contribution (CIR-SV)

Compared to Figure 1, the distribution of the considered funds are generally less "concentrated" and a summary of some estimated percentiles in Table 5 further indicates "fatter" tails when compared to the (BS) model.

Strategy	5%	25%	Median	75%	95%
Balanced A/B	−2.99%	1.34%	4.09%	6.81%	10.64%
Life-cycle A/B	−2.98%	1.28%	4.09%	6.90%	10.68%
Balanced C	−4.66%	0.49%	3.76%	7.01%	11.63%
Life-cycle C	−4.68%	0.40%	3.75%	6.98%	11.38%

Table 5: Key figures – single contribution (CIR-SV)

However, our main focus is not on analyzing the effects of the change of model on the observed returns but rather on investigating whether the calibrated balanced funds still provide an accurate approximation for the life-cycle funds. Figure 2 and Table 5 indicate that the distributions of the life-cycle funds are very similar to their respective approximating balanced funds. We performed a two-sample Kolmogorov-Smirnov and a two-sample

Anderson-Darling test with the null-hypothesis that the above empirical distributions were drawn from the same origin distributions. Assuming a significance level of 5%, the null-hypothesis was not rejected.[12] Obviously, this is not a proof that they actually have the same underlying probability distributions but at least a good indicator that the distributions do not differ too much. However, since the percentiles shown in Table 5 are "just" point estimates after a simulation of one "bucket" of 20,000 trajectories, Appendix C shows the percentiles' confidence intervals with a level of 95% after applying a more in depth Monte-Carlo study. Following these results, we conclude from Table 12 in Appendix C, that e.g. considering the 5^{th} percentile of above distributions a difference of roughly 0.2% for strategy A/B and 0.1% for strategy C with their corresponding balanced funds' 5^{th} percentiles is likely. In summary, we can conclude that the balanced funds appropriately approximate a very large part of the life-cycle funds' risk-return profiles.

The (CIR-SVJD) model additionally allows for jumps in equity returns. The corresponding results are shown in Figure 3 and Table 6.

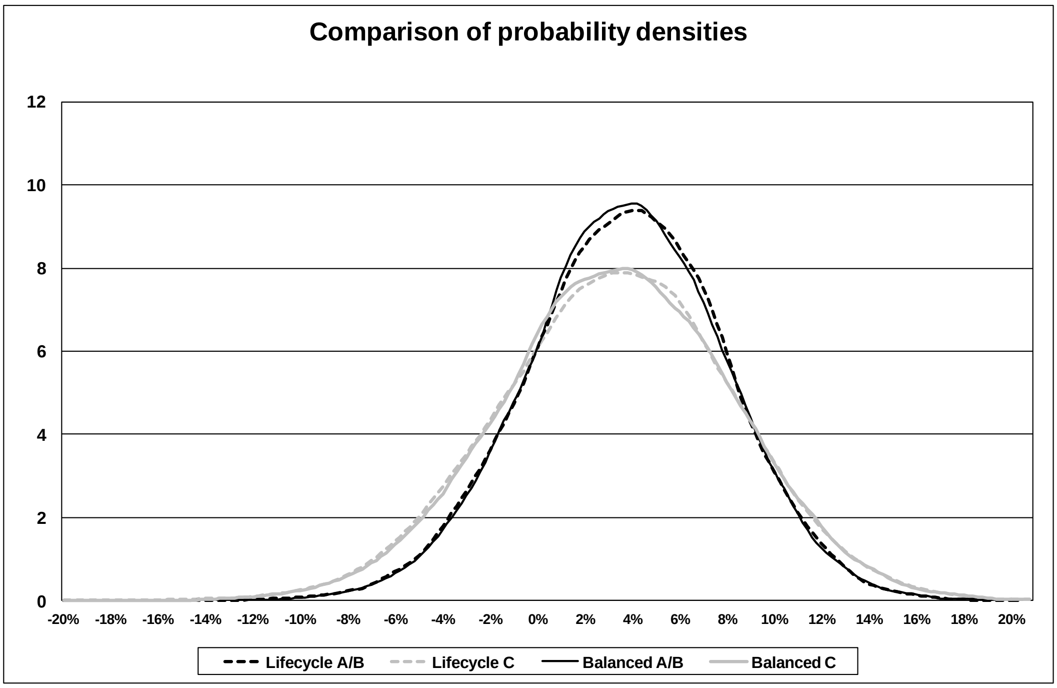

Figure 3: Life-cycle vs. balanced fund – single contribution (CIR-SVJD)

[12] The results of the test statistics are summarized in Appendix B.

Again, the distributions of life-cycle and balanced funds' returns in Figure 3 look very similar. Hence, the matching balanced funds derived in the simple (BS) model allow for a good assessment of the life-cycle funds' risk-return profiles..

Strategy	**5%**	**25%**	**Median**	**75%**	**95%**
Balanced A/B	−3.03%	1.14%	3.89%	6.71%	10.63%
Life-cycle A/B	−3.17%	1.09%	3.96%	6.75%	10.67%
Balanced C	−4.74%	0.22%	3.52%	6.88%	11.62%
Life-cycle C	−4.88%	0.14%	3.54%	6.84%	11.57%

Table 6: Key figures – single contribution (CIR-SVJD)

From Table 6, we can see that as expected, the additional introduction of jumps negatively affects the returns as compared to the previous (CIR-SV) model. However, the returns are not tremendously changed due to the rather long investment horizon and the relatively small (expected) jump sizes (cf. Table 2). Further, the impact of adding jumps to the equity process seems to affect balanced and life-cycle funds in a similar way and hence the distributions still appear reasonably similar. The null-hypothesis that the observed returns were drawn from the same origin distribution is again not rejected (cf. Appendix B) although estimated p-Values generally decrease. In line with the results for the (CIR-SV) model, some minor difference of the resulting risk-return profiles of life-cycle and corresponding balanced funds is likely, e.g. when the confidence intervals for their 5^{th} percentile are considered (cf. Table 13 in Appendix C).

Summarizing, the balanced funds derived by Proposition 1 provide accurate approximations of the considered life-cycle funds' risk-return profiles even under more complex financial models and hence seem an appropriate tool for quickly assessing risk-return profiles of life-cycle funds. Further, the very existence or additional value of life-cycle investment is challenged when single contributions are considered and only returns at maturity are relevant.

Our results also have serious policy implications: We show that life-cycle funds yield the same result as a suitably calibrated (and much simpler) balanced fund. Moreover, if this balanced fund is equally expensive as the life-cycle fund, it even dominates the risk-return profile of the life-cycle fund. Hence, using life-cycle funds as a default choice should in our opinion be reconsidered by pension plan managers since it adds complexity and reduces expected return.

5 Regular contributions

In this section we will analyze the case of regular contributions. We again start in a (BS) economy in Section 5.1 and then switch to more complex asset models in Section 5.2.

5.1 Matching a balanced fund in the (BS) model

In contrast to the single contribution case, analytical solutions for the considered probability distributions do not exist for regular contributions even in the simple (BS) economy.[13] Therefore, we rely on a moment matching procedure to calibrate the balanced funds under consideration, i.e. we determine the balanced fund's equity portion x_S and its management fee c_{BF} such that the first and second moment of the resulting distribution of wealth coincide with the corresponding moments of the life-cycle fund at maturity T. For ease of notation and without loss of generality we assume a yearly[14] contribution of 1 unit of currency.

Let $W_{LF}(T)$ denote the wealth at time T obtained after annually contributing to the life-cycle fund. Hence, we obtain $W_{LF}(T) = \sum_{i=0}^{T-1} \frac{V_{LF}(T)}{V_{LF}(i)} = \frac{V_{LF}(T)}{V_{LF}(T-1)} \cdot \left(1 + W_{LF}(T-1)\right)$ which can be calculated recursively applying $W_{LF}(1) = \frac{V_{LF}(1)}{V_{LF}(0)}$.

Further, let $Z_{LF}(t) \coloneqq \frac{V_{LF}(t)}{V_{LF}(t-1)}$ denote the annual returns of the life-cycle fund. Assuming a (BS) economy, we obtain $Z_{LF}(t) \sim \mathcal{LN}\left(\mu_{LF}(t), \sigma_{LF}^2(t)\right)$ with $\mu_{LF}(t) = \int_{t-1}^{t} \left(x_{S,u}(r + \lambda_S) + \left(1 - x_{S,u}\right)r - c_{LF} - \frac{1}{2}\left(x_{S,u}\sigma_S\right)^2\right) du$ and $\sigma_{LF}^2(t) = \int_{t-1}^{t} \left(x_{S,u}\sigma_S\right)^2$ since $Z_{LF}(t) = \exp\left(\int_{t-1}^{t} \left(x_{S,u}(r + \lambda_S) + \left(1 - x_{S,u}\right)r - c_{LF} - \frac{1}{2}\left(x_{S,u}\sigma_S\right)^2\right) du + \int_{t-1}^{t} \left(x_{S,u}\sigma_S\right) dW_1(u)\right)$. In

[13] When regular contributions are considered, essentially a treatment of a sum of log-normal distributed random variables is necessary. E.g. Dufresne (2004) provides an analysis of different approximations of this type of random variable.

[14] The same ideas can be applied to arbitrary regular contributions such as quarterly or monthly premium contributions.

addition, $Z_{LF}(1), \dots, Z_{LF}(T)$ are stochastically independent random variables due to the independent increments of the underlying Brownian Motion.

Therefore, the first moment $\mathbb{E}_{\mathcal{P}} W_{LF}(T)$ is derived as $\mathbb{E}_{\mathcal{P}} W_{LF}(T) = \mathbb{E}_{\mathcal{P}} Z_{LF}(T) \cdot \big(1 + \mathbb{E}_{\mathcal{P}} W_{LF}(T-1)\big)$ and recursively calculated using $\mathbb{E}_{\mathcal{P}} W_{LF}(1) = \mathbb{E}_{\mathcal{P}} Z_{LF}(1)$. Further, the second moment $\mathbb{E}_{\mathcal{P}} W_{LF}^2(T)$ is given by $\mathbb{E}_{\mathcal{P}} W_{LF}^2(T) = \mathbb{E}_{\mathcal{P}} Z_{LF}^2(T) \cdot \big(1 + 2\mathbb{E}_{\mathcal{P}} W_{LF}(T-1) + \mathbb{E}_{\mathcal{P}} W_{LF}^2(T-1)\big)$ which is again determined recursively using $\mathbb{E}_{\mathcal{P}} W_{LF}^2(1) = \mathbb{E}_{\mathcal{P}} Z_{LF}^2(1)$. Finally, the first and second moment of the distribution of wealth when investing annually in a balanced fund are derived similarly. This yields the following proposition for regular contributions in a (BS) model.

PROPOSITION 2

Consider regular contributions and an investment horizon of T years for an arbitrary life-cycle fund characterized by its glide-path $\left(x_{S,t}\right)_t$ and its management fee c_{LF}. If, for the solution (x_0, y_0) of a corresponding two-dimensional polynomial equation set, the conditions $y_0 \leq x_0^2 \exp(\sigma_S^2)$ and $x_0 \leq \exp\left(r + \lambda_S \sqrt{\frac{\log y_0 - 2\log x_0}{\sigma_S^2}}\right)$ hold, a unique balanced fund with $x_S \in [0,1]$ and $c_{BF} \geq 0$ exists that exactly matches the first and second moment of the life-cycle fund's distribution of wealth at maturity T.

Proof:

see Appendix A.

In this appendix, we first show that deriving a balanced fund to match the first and second moment of life-cycle fund's distribution of wealth at maturity T is closely related to solving a two-dimensional polynomial equation set. We further show that unique solutions – say x_0 and y_0– to this problem exist. If the above conditions hold, we can derive a valid balanced fund – that is $x_S \in [0,1]$ and $c_{BF} \geq 0$ – from x_0 and y_0.

□

Consequences of Proposition 2

In case of regular contributions and if the above conditions hold, there exists a balanced fund which yields the same expected return and variance of wealth at maturity as a given life-cycle fund. Further, although no analytical solution of the distribution of $W_{LF}(T)$ is available, the moments and hence the calibrated balanced fund can be derived analytically.

The following analyses show that the moment-matching balanced fund closely matches the risk-return profile of the corresponding life-cycle fund. As shown in Appendix A, calibrating the balanced fund is essentially equivalent with solving a polynomial equation set which then boils down to solving for roots of a polynomial.[15] Applying this methodology, Table 7 displays the balanced funds calibrated to the different life-cycle strategies A, B and C as introduced in Section 2. Similar with Section 4, we set the life-cycle funds' management fee to $c_{LF} = 1.3\%\ p.a.$, apply the (BS) model as parameterized in Section 3 and assume an investment horizon of $T = 12$ years. In addition, Appendix D provides even more numerical examples on different life-cycle strategies and their corresponding balanced fund counterparts.

Strategy	**Equity portion x_S**	**Management fee c_{BF}**
A	35.88%	$1.32\%\ p.a.$
B	76.08%	$1.63\%\ p.a.$
C	67.25%	$1.91\%\ p.a.$

Table 7: Balanced funds' calibration – regular contributions

Comparing the regular with the single contribution case (cf. Table 3) shows a massive impact of the premium payment mode on the calibrated balanced funds and hence on the risk-return profiles of the considered life-cycle funds. In contrast to the single contribution setting where life-cycle strategy A and B resulted in exactly the same risk-return profile and hence were matched with the same balanced fund, the classical life-cycle strategy A is now associated

[15] Standard numerical packages contain solvers for deriving polynomial's roots, most of them relying on versions of an algorithm introduced by Jenkins and Traub (1970).

with a far more conservative balanced fund than strategy B. This results from the amount of capital that is exposed to the risky asset: Strategy A decreases its equity exposure over time. Hence, the more capital has been invested, the less risky it is allocated. Strategy B does the exact opposite and is now even more aggressive than strategy C which was the "riskiest" life-cycle strategy in the single contribution case. Further it is worth noting that similar with the results in Section 4, all balanced funds require higher management fees than the considered life-cycle funds.

We now analyze the quality of the approximation through the matching balanced funds. Figure 4 and Table 8 summarize the empirical distribution of internal rates of return obtained by applying 20,000 Monte-Carlo trajectories assuming the (BS) model. We additionally compare a balanced fund with equity portion $x_S = 50\%$ and $c_{BF} = c_{LF} = 1.3\%\, p.a.$ as derived by the rule-of-thumb calibration technique which does not distinguish between single and regular contributions.

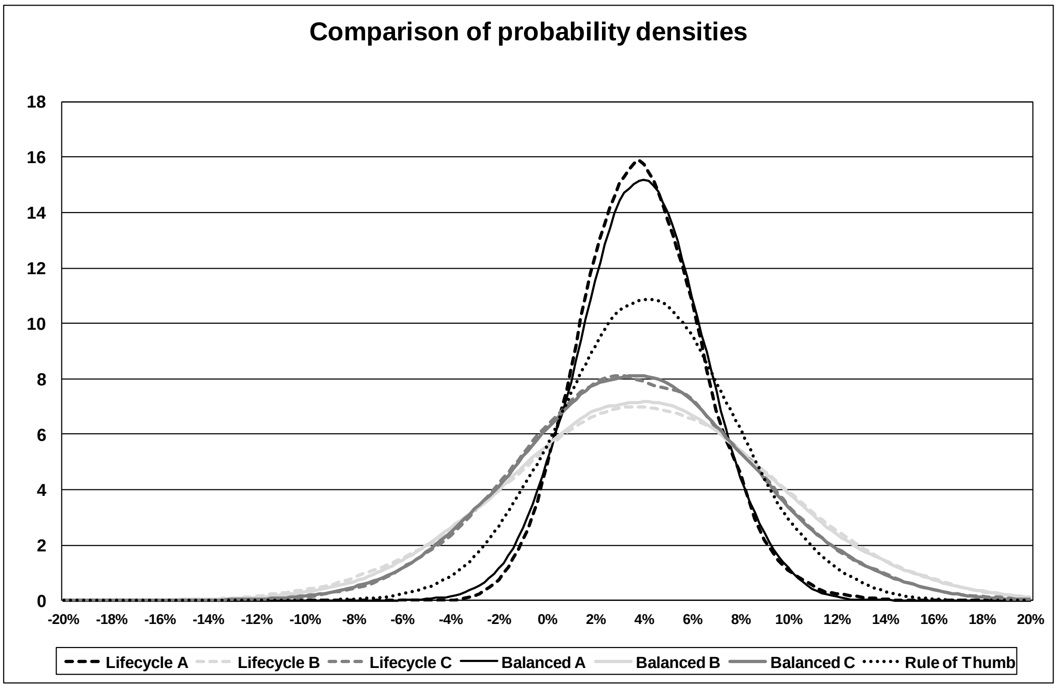

Figure 4: Life-cycle vs. balanced fund – regular contributions (BS)

The rule of thumb approximation now completely fails to approximate the considered life-cycle funds' return distributions and is hence of only very limited explanatory value to financial advisors and their clients. Further, the calibrated balanced funds seem to deliver appropriate and accurate approximations for the life-cycle funds' risk-return profiles.

However, Table 8 shows some deviations, e.g. the balanced funds' 5^{th} percentile under (over-) estimates the life-cycle funds' 5^{th} percentile for strategy A (B).

Strategy	**5%**	**25%**	**Median**	**75%**	**95%**
Balanced A	−0.26%	2.31%	4.06%	5.80%	8.42%
Life-cycle A	0.08%	2.33%	4.00%	5.75%	8.40%
Balanced B	−5.04%	0.34%	4.05%	7.76%	13.37%
Life-cycle B	−5.38%	0.24%	4.06%	7.85%	13.39%
Balanced C	−4.29%	0.46%	3.74%	7.01%	11.95%
Life-cycle C	−4.15%	0.48%	3.71%	7.05%	11.96%

Table 8: Key figures – regular contributions (BS)

Hence, although the first two moments of the balanced funds and the corresponding life-cycle funds coincide, the balanced funds do not exactly match the life-cycle funds' risk-return profiles. In this case, for strategy A the null hypothesis is rejected for a confidence level of 5% mainly due to the difference of the probability distributions in the lower tail (see Appendix B). In addition, comparing above point estimates for the different percentiles with their confidence intervals (cf. Table 14 in Appendix C) further yields some (minor) difference in upper percentiles (e.g. 75^{th} or 95^{th}) as well.

Roughly speaking, the balanced fund under- (over-) estimates the lower tail of the life-cycle fund's distribution if the life-cycle fund's glide path decreases (increases) its equity exposure over time. Since strategy C applies a permanent reallocation of risky and riskless asset, both effects "cancel out" partly and hence the estimate of the balanced fund's lower tail is more close to the life-cycle fund's lower tail.

Bearing these limitations in mind, the introduced approach still supports financial advisors and clients since it allows for an easy assessment of the risk and upside potential of life-cycle funds by just considering the respective matching balanced funds. Moreover, similar to the single contribution case, the additional value of investing in a life-cycle strategy is challenged, in particular when taking into account that matching balanced funds would require higher management fees (cf. Table 7).

In other words: If the balanced funds were equipped with the life-cycle funds' management fee (1.3% $p.a.$ in our example) the expected return would exceed the life-cycle funds' expected returns. This is of particular relevance since it should be possible to manage the much simpler balanced funds at the same (or even lower) cost as life-cycle funds. Hence, even in the setting of regular contributions balanced funds may stochastically dominate the life-cycle fund investment if only returns at maturity are considered.

5.2 Model risk

As in Section 4.3, we will now analyze the accuracy of the approximation under more complex asset models. In what follows, we only display the results of the (CIR-SVJD) model since for the (CIR-SV) model similar results (in terms of accuracy of the approximation) are obtained. Again note that the balanced funds calibrated according to proposition 2 and summarized in Table 7 are analyzed.

Figure 5 shows the empirical returns of the considered life-cycle funds and their corresponding balanced funds assuming the (CIR-SVJD) model as introduced in Section 3 applying 20,000 Monte-Carlo trajectories.

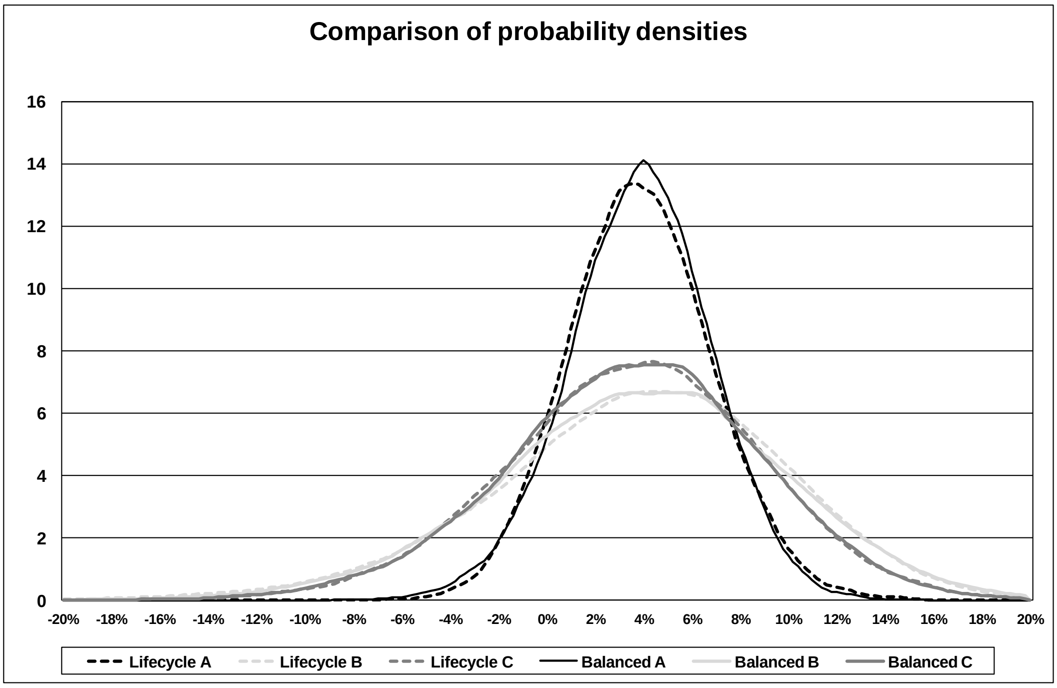

Figure 5: Life-cycle vs. balanced fund – regular contributions (CIR-SVJD)

Again, the empirical densities look very similar and the balanced funds seem to appropriately approximate the life-cycle funds' risk-return profile. However, similar with the results in the (BS) model, the distributions do not entirely coincide which can also be seen in Table 9 and is further confirmed by the test statistics in Appendix B where for strategies A and B the null hypothesis is rejected. [16]

[16] The corresponding confidence intervals summarized in Table 15 in Appendix C also (naturally) detect the differences.

Strategy	5%	25%	Median	75%	95%
Balanced A	−1.00%	2.05%	4.04%	5.95%	8.75%
Life-cycle A	−0.81%	1.95%	3.92%	5.94%	9.02%
Balanced B	−6.19%	0.11%	4.12%	7.99%	13.55%
Life-cycle B	−6.74%	0.13%	4.26%	8.11%	13.36%
Balanced C	−5.29%	0.24%	3.79%	7.22%	12.14%
Life-cycle C	−5.28%	0.22%	3.81%	7.25%	12.08%

Table 9: Key figures – regular contributions (CIR-SVJD)

The discrepancies between the lower percentiles of the life-cycle and their corresponding balanced funds are similar as in the (BS) model. Further, the upper percentiles (e.g. the 95th-percentile) now also show some differences. In line with the (BS) model, the 5th- and 95th-percentile are roughly (under-) overestimated when a decreasing (increasing) equity share in the life-cycle funds' glide path is applied.

In summary, the calibration algorithm that is based on a moment matching procedure is able to capture a major part (e.g. 5th–95th-percentile) of the considered life-cycle funds' return distributions, however fails in exactly reproducing the tails of the underlying life-cycle investments. Therefore, the approximation's appropriateness from a statistical point of view is rejected. These shortcomings might however be acceptable from a practitioner's point of view, in particular since the approximation is clearly superior to the previously proposed rule of thumb. Further, the very existence of life-cycle funds is challenged, at least for buy and hold investors. Therefore, even considering the case of regular contributions, the choice of life-cycle funds as a default investment option might need to be revisited.

6 Conclusion and outlook

In this paper we have analyzed the risk-return profiles – i.e. the probability distribution of returns at the end of the investment horizon – of life-cycle funds and balanced funds. In contrast to previous research, we have not focused on comparing the return distribution of more or less arbitrarily selected life-cycle and balanced funds, but provided methodologies to construct balanced funds matching or dominating the risk-return profile of any given life-cycle fund.

For a single contribution in a Black-Scholes economy, we have provided closed form solutions for balanced funds exactly matching the risk-return profile of any given life-cycle fund. We have further identified that a balanced fund exists that stochastically dominates the given life-cycle fund. Further, applying Monte-Carlo techniques we have shown that – even when more complex asset models are considered – the approximations derived in a Black-Scholes economy deliver appropriate results and statistical tests detect no difference between the underlying risk-return profiles.

Regarding regular contributions and assuming a Black-Scholes economy, we have shown that a unique balanced fund exists matching the first two moments of the wealth distribution for any given life-cycle fund if some additional conditions are fulfilled. These approximations were able to explain a major part of the life-cycle fund's risk-return profile even under more complex asset models. However, the quality of the approximation is somewhat lower in the tails and hence the applied statistical tests rejected the "equality" of the corresponding distributions. Nevertheless, the approximations still seem useful from a practical point of view, since the difference was of minor magnitude especially when compared to rule of thumb approximations previously proposed in the literature and used by practitioners so far. In addition, potentially dominating balanced funds may be constructed as well.

Therefore, our results on the one hand facilitate financial planning by allowing for an easy assessment of the risk-return profile of "complex" life-cycle funds by their less complex balanced fund counterparts. On the other hand, the results challenge the very existence of (the considered types of) life-cycle funds since balanced funds delivering a very similar risk-return profile or even dominating the return distribution are available.

Our results should therefore be of great interest to pension plan providers and regulators. The choice of life-cycle funds as a default investment option is seriously questioned by our analyses. Although life-cycle funds might be perceived as a "safe" investment strategy (due to the lower risk close to retirement), they bear the same risk exposure as a suitably calibrated (and much simpler) balanced funds. Moreover, if a corresponding balanced fund is equally expensive as the life-cycle fund, it often dominates the risk-return profile of the life-cycle fund. Hence, life-cycle funds add complexity while at the same time reducing expected return (at a given level of risk).

Of course, our research allows for refinement and amendments. The approximations may be derived applying a Black-Scholes model with time-dependant drift in order to match a given term structure of interest rates at outset. Further, challenging the approximations' appropriateness using some historical data e.g. within a Bootstrap approach seems worthwhile. Finally, our findings were based on the investment vehicles' returns at maturity neglecting what happens during the investment phase. In particular in the case of regular contributions and for investors who do not only focus on the utility derived from the maturity value of their investment, there might be reasons why life-cycle funds could be preferred over balanced funds.

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A Matching for regular contributions – Proof of proposition 2

In order to proof proposition 2, we will investigate the matching of first and second moment of the wealth distribution in the regular premium case. We consider a Black-Scholes model as defined in Section 3 and further assume annual contributions of 1 unit of currency to some life-cycle/balanced fund.

For $t = 1, \dots T$ let $Z(t)$ denote independent copies of a log-normal distributed random variable $Z \sim \mathcal{LN}(\mu, \sigma^2)$. In what follows – assuming an annual contribution of 1 unit of currency to an investment delivering returns equal to the log-normal distributed random variables $Z(t)$ – we will show how to derive μ and σ^2 such that first and second moment of this investment and an annual contribution to some arbitrary life-cycle fund coincide. Then, we will state sufficient conditions on μ and σ^2 such that a valid[17] balanced fund can be derived from μ and σ^2 accordingly. Note, if one defines a balanced fund with some equity portion x_S and some management fee c_{BF}, one obtains $\frac{V_{BF}(t)}{V_{BF}(t-1)} \sim \mathcal{LN}(\mu_{BF}, \sigma_{BF}^2)$ with $\mu_{BF} = r + x_S\lambda_S - c_{BF} - \frac{1}{2}(x_S\sigma_S)^2$ and $\sigma_{BF}^2 = (x_S\sigma_s)^2$.

The proof of proposition 2 is now split in three parts: First, we state the required calibration procedure as a problem of solving a polynomial equation set. Second, we derive sufficient conditions to obtain a unique and valid balanced fund to solve these equations and then close the analysis with a discussion of the sufficient conditions.

Solving a polynomial equation set to calibrate first and second moment

In what follows, let $W(T) = \sum_{i=0}^{T-1} \frac{V(T)}{V(i)} = \frac{V(T)}{V(T-1)} \cdot \left(1 + W(T-1)\right)$ denote the outcome of an annual investment of 1 unit of currency in the underlying copies of Z, i.e, set $V(t) \coloneqq \prod_{i=1}^{t} Z(t)$ and further set $x \coloneqq \mathbb{E}_{\mathcal{P}} Z = \exp\left(\mu + \frac{1}{2}\sigma^2\right)$ and $y \coloneqq \mathbb{E}_{\mathcal{P}} Z^2 = \exp(2\mu + 2\sigma^2)$. It is easily shown (e.g. by induction) that $\mathbb{E}_{\mathcal{P}} W(T) = \sum_{j=1}^{T} x^j$ holds. Further, the second moment of $W(T)$ is characterized by the following lemma.

[17] i.e. solutions with "admissible" equity portion (not less than 0% and not greater than 100%) and admissible management fee not less than 0% p.a.

LEMMA 1

$$\mathbb{E}_{\mathcal{P}} W^2(T) = y^T + \sum_{i=1}^{T-1} \underbrace{\left(2\sum_{j=1}^{T-i} x^j + 1\right)}_{:=a_i(x)} y^i$$

Proof by induction:

$T = 1$

$\mathbb{E}_{\mathcal{P}} W^2(1) = \mathbb{E}_{\mathcal{P}} Z^2(1) = y$ holds by definition.

$T - 1 \rightarrow T$

Similar calculus as performed in Section 5 gives (applying the induction hypothesis)

$$\begin{aligned}
&\mathbb{E}_{\mathcal{P}} W^2(T) = y\big(1 + 2\mathbb{E}_{\mathcal{P}} W(T-1) + \mathbb{E}_{\mathcal{P}} W^2(T-1)\big) \\
&= \left(2\sum_{j=1}^{T-1} x^j + 1\right) y + y \cdot \left(y^{T-1} + \sum_{i=1}^{T-2}\left(2\sum_{j=1}^{T-1-i} x^j + 1\right) y^i\right) \\
&= \left(2\sum_{j=1}^{T-1} x^j + 1\right) y + y^T + \sum_{i=1}^{T-2}\left(2\sum_{j=1}^{T-1-i} x^j + 1\right) y^{i+1} \\
&= \left(2\sum_{j=1}^{T-1} x^j + 1\right) y + y^T + \sum_{i=2}^{T-1}\left(2\sum_{j=1}^{T-1-(i-1)} x^j + 1\right) y^i \\
&= y^T + \textstyle\sum_{i=1}^{T-1}\big(2\sum_{j=1}^{T-i} x^j + 1\big) y^i.
\end{aligned}$$

□

Therefore, the first and second moment of $W(T)$ match the first and second moment of $W_{LF}(T)$ if (real) roots of $f_T(x)$ and $g_{x,T}(y)$ defined as follows can be computed.

$$f_T(x) := \sum_{i=1}^{T} x^i - \mathbb{E}_{\mathcal{P}} W_{LF}(T)$$

$$g_{x,T}(y) := y^T + \sum_{i=1}^{T-1} \left(2 \sum_{j=1}^{T-i} x^j + 1 \right) y^i - \mathbb{E}_{\mathcal{P}} W_{LF}^2(T)$$

If real roots (e.g. x_0 and y_0) were found, we obtained a log-normal distribution Z with μ and σ^2 from which (possibly) a balanced fund's equity portion x_S and its management fee c_{BF} can be derived. However, it is not yet guaranteed if x_S and c_{BF} indeed define a "valid" balanced fund, that is $0\% \leq x_S \leq 100\%$ and $c_{BF} \geq 0\%\ p.a.$

In what follows, we will therefore first prove the (unique) existence of above roots and provide sufficient conditions for the validity of the derived balanced fund from these roots.

Sufficient conditions for obtaining unique and valid balanced funds

LEMMA 2

Unique positive real roots x_0 and y_0 of $f_T(x)$ and $g_{x_{0},T}(y)$ exist.

Proof:

It is easily shown that $\mathbb{E}_{\mathcal{P}} W_{LF}(T) > 0$ and $\mathbb{E}_{\mathcal{P}} W_{LF}^2\,(T) > 0$ holds. We further obtain $f_T(0) = -\mathbb{E}_{\mathcal{P}} W_{LF}(T) < 0$ and $\lim_{x\to\infty} f_T(x) = \infty$ and therefore (at least) one real solution $x_0 > 0$ with $f_T(x_0) = 0$ exists applying the intermediate value theorem. Similar arguments yield to (at least one) real solution $y_0 > 0$ with $g_{x_{0},T}(y_0) = 0$. Further, Descartes' rule of signs gives at most one positive real solution for the polynomials considered.[18] Therefore, the solutions x_0 and y_0 are unique.

$\square$

[18] Published by René Descartes in his work "La Géometrie" in 1637 and often revisited nowadays e.g. in Anderson et al. (1998).

Although x_0 and y_0 guarantee to match the first moments of the life-cycle fund, it is not yet ensured if x_0 and y_0 are actually moments of a valid balanced fund or even generated by a valid lognormal distributed random variable. Note, for defining a valid log-normal distributed random variable $\sigma^2 > 0$ is essentially required in this setting. In the next lemma, we propose sufficient conditions for x_0 and y_0 actually being generated by a valid balanced fund.

LEMMA 3

If $y_0 > x_0^2$, $y_0 \leq x_0^2 \exp(\sigma_S^2)$ and $x_0 \leq \exp\left(r + \lambda_S \sqrt{\frac{\log y_0 - 2\log x_0}{\sigma_S^2}}\right)$, then the balanced fund equipped with equity portion $x_S \coloneqq \sqrt{\frac{\log y_0 - 2\log x_0}{\sigma_S^2}}$ and management fee $c_{BF} \coloneqq r + \lambda_S x_S - \log x_0$ is valid. Further, an annual contribution of 1 unit of currency into this balanced fund yields to the same first and second moment of annually contributing to the considered life-cycle fund.

Proof:

Solving x_0 and y_0 for μ and σ^2 gives $\mu = \log x_0 - \frac{1}{2}\sigma^2$ and $\sigma^2 = \log y_0 - 2\log x_0$. Hence, if $y_0 > x_0^2$ holds, we obtain $\sigma^2 > 0$ and thus in first instance a valid lognormal distribution.

Now, consider a balanced fund with equity portion $x_S \coloneqq \sqrt{\frac{\sigma^2}{\sigma_S^2}}$ and management fee $c_{BF} \coloneqq r + \lambda_S x_S - \log x_0$. Then, we obtain

$$0 \leq x_S = \sqrt{\frac{\sigma^2}{\sigma_S^2}} = \sqrt{\frac{\log y_0 - 2\log x_0}{\sigma_S^2}} \leq \sqrt{\frac{\log(x_0^2 \exp(\sigma_S^2)) - 2\log x_0}{\sigma_S^2}} = 1$$

$$c_{BF} \geq r + \lambda_S x_S - \left(r + \lambda_S \sqrt{\frac{\sigma^2}{\sigma_S^2}}\right) = 0$$

and thus indeed a valid balanced fund.

Further, this balanced fund is log-normal distributed (since we are positioned in a (BS) model) and we get $\mu_{BF} = r + x_S\lambda_S - c_{BF} - \frac{1}{2}(x_S\sigma_S)^2 = \log x_0 - \frac{1}{2}\sigma^2 = \mu$ and $\sigma_{BF}^2 = (x_S\sigma_s)^2 = \sigma^2$. Hence, the balanced fund's returns are similarly distributed as the initial log-normal random variable Z we started with and therefore (by construction of x_0 and y_0) yields to the required solution.

$\square$

Some thoughts on the sufficient conditions

We start with investigating the condition $y_0 > x_0^2$ which can be shown to be fulfilled in general. For the following analysis, let $\tilde{f}_T(x) \coloneqq f_T(x) + \mathbb{E}_{\mathcal{P}} W_{LF}(T)$ and $\tilde{g}_{x,T}(y) \coloneqq g_{x,T}(y) + \mathbb{E}_{\mathcal{P}} W_{LF}^2(T)$.

Note that $\tilde{g}'_{x,T}(y) > 0 \; \forall y > 0$ for any fixed $x > 0$, i.e. $\tilde{g}_{x,T}(y)$ is monotonically increasing for positive x and y. Hence, for ensuring $y_0 > x_0^2$ it is sufficient to show $\tilde{g}_{x_0,T}(y_0) > \tilde{g}_{x_0,T}(x_0^2)$. First, we show $\tilde{g}_{x,T}(x^2) = \tilde{f}_T^2(x)$ again using induction.

$T = 1$

$$\tilde{f}_1^2(x) = x^2 = \tilde{g}_{x,1}(x)$$

$T - 1 \to T$

$\tilde{g}_{x,T}(x^2)$ is written as

$$\begin{aligned} \tilde{g}_{x,T}(x^2) &= x^{2T} + \sum_{i=1}^{T-1} \left(2 \sum_{i=1}^{T-i} x^j + 1 \right) x^{2i} \\ &= x^{2T} + \sum_{i=1}^{T-1} x^{2i} + 2 \sum_{i=1}^{T-1} x^{2i} \sum_{j=1}^{T-i} x^j \end{aligned}$$

Using the induction hypothesis then gives

$$\begin{aligned} \tilde{f}_T^2(x) = \left(\sum_{i=1}^{T} x^i \right)^2 &= x^{2T} + 2x^T \sum_{i=1}^{T-1} x^i + \left(\sum_{i=1}^{T-1} x^i \right)^2 \\ &= x^{2T} + 2x^T \sum_{i=1}^{T-1} x^i + x^{2(T-1)} + \sum_{i=1}^{T-2} \left(2 \sum_{j=1}^{T-1-j} x^j + 1 \right) x^{2i} \end{aligned}$$

$$= x^{2T} + 2x^T \sum_{i=1}^{T-1} x^i + x^{2(T-1)} + 2 \sum_{i=1}^{T-2} \sum_{j=1}^{T-1-j} x^{j+2i} + \sum_{i=1}^{T-2} x^{2i}$$

$$= x^{2T} + \sum_{i=1}^{T-1} x^{2i} + 2x^T \sum_{i=1}^{T-1} x^i + 2 \sum_{i=1}^{T-2} x^{2i} \sum_{j=1}^{T-1-j} x^j$$

Fairly simple algebra (e.g. using Geometric row expansion) then yields $\sum_{i=1}^{T-1} x^{2i} \sum_{j=1}^{T-i} x^j = x^T \sum_{i=1}^{T-1} x^i + \sum_{i=1}^{T-2} x^{2i} \sum_{j=1}^{T-1-j} x^j$ which completes the proof.

If the life-cycle fund's wealth distribution $W_{LF}(T)$ further allows for a positive variance[19], we finally obtain $\tilde{g}_{x_0,T}(y_0) = \mathbb{E}_{\mathcal{P}}\left(W_{LF}^2(T)\right) > \mathbb{E}_{\mathcal{P}}\left(W_{LF}(T)\right)^2 = \tilde{f}_T^2(x_0) = \tilde{g}_{x_0,T}(x_0^2)$ and hence $y_0 > x_0^2$.

Unfortunately, we are not able to show if the conditions $y_0 \leq x_0^2 \exp(\sigma_S^2)$ and $x_0 \leq \exp\left(r + \lambda_S \sqrt{\frac{\log y_0 - 2\log x_0}{\sigma_S^2}}\right)$ are fulfilled in general. However, our numerical analyses in Appendix D indicate that they at least hold for a broad type of different assumptions on the life-cycle's glide path. Hence, there is some indication that these conditions may hold in general.

B Results of statistical tests

This section summarizes the statistical tests – especially their p-values – we performed within our analyses. We used the computational package R (2010) and further RExcel as developed by Baier and Neuwirth (2007). Results for the single contribution case are given in Table 10 whereas results in the regular contribution case are summarized in Table 11.

[19] which in the (BS) model is fulfilled when the underlying asset strategy does not consist of a pure investment in the risk-free asset only.

Capital market model	**Strategy**	**Kolmogorov-Smirnov**	**Anderson-Darling**
CIR-SV	A	0.6609	0.4242
	C	0.7842	0.3521
CIR-SVJD	A	0.2921	0.3043
	C	0.3592	0.3533

Table 10: Test for equality of distributions – single contribution

Capital market model	**Strategy**	**Kolmogorov-Smirnov**	**Anderson-Darling**
BS	A	0.018	0.000
	B	0.359	0.104
	C	0.890	0.478
CIR-SVJD	A	0.000	0.000
	B	0.149	0.015
	C	0.830	0.613

Table 11: Test for equality of distributions – regular contributions

C Confidence intervals for point estimates

This section gives confidence levels for the point estimates of different percentiles estimated by means of Monte-Carlo simulation in Sections 4.2 and 5.

For deriving the following 95% confidence intervals, we repeated the Monte-Carlo analyses of projecting buckets of 20,000 trajectories 200 times and then estimated the respective point estimates' mean (say μ_{perc}) and its standard error (say σ_{perc}). The confidence intervals were then derived by $[\mu_{perc} - z_{95\%} \cdot \sigma_{perc}, \mu_{perc} + z_{95\%} \cdot \sigma_{perc}]$ where $z_{95\%}$ gives the 95th percentile of the standard normal distribution. Following tables show the mean of the different percentiles obtained in *italic* font after repeating the Monte-Carlo procedure and further give the corresponding confidence intervals.

Single contribution

Strategy	**5%**	**25%**	**Median**	**75%**	**95%**
Balanced A/B	*-2.87%*	*1.28%*	*4.06%*	*6.77%*	*10.59%*
	(-2.97% , -2.76%)	(1.21% , 1.35%)	(4.00% , 4.12%)	(6.70% , 6.84%)	(10.49% , 10.68%)
Life-cycle A/B	*-3.07%*	*1.24%*	*4.07%*	*6.81%*	*10.63%*
	(-3.18% , -2.96%)	(1.17% , 1.30%)	(4.01% , 4.13%)	(6.74% , 6.87%)	(10.54% , 10.72%)
Balanced C	*-4.51%*	*0.41%*	*3.72%*	*6.97%*	*11.56%*
	(-4.63% , -4.39%)	(0.33% , 0.49%)	(3.65% , 3.79%)	(6.89% , 7.05%)	(11.45% , 11.67%)
Life-cycle C	*-4.64%*	*0.37%*	*3.70%*	*6.94%*	*11.49%*
	(-4.76% , -4.52%)	(0.28% , 0.45%)	(3.62% , 3.78%)	(6.86% , 7.02%)	(11.37% , 11.60%)

Table 12: Confidence intervals – single contribution (CIR-SV)

Strategy	5%	25%	Median	75%	95%
Balanced A/B	*-3.08%*	*1.12%*	*3.96%*	*6.73%*	*10.66%*
	(-3.18% , -2.98%)	(1.05% , 1.19%)	(3.89% , 4.02%)	(6.66% , 6.79%)	(10.56% , 10.76%)
Life-cycle A/B	*-3.27%*	*1.09%*	*3.97%*	*6.77%*	*10.71%*
	(-3.38% , -3.16%)	(1.02% , 1.16%)	(3.91% , 4.04%)	(6.70% , 6.84%)	(10.61% , 10.81%)
Balanced C	*-4.76%*	*0.21%*	*3.59%*	*6.91%*	*11.63%*
	(-4.89% , -4.64%)	(0.13% , 0.29%)	(3.51% , 3.67%)	(6.83% , 6.99%)	(11.51% , 11.76%)
Life-cycle C	*-4.88%*	*0.19%*	*3.59%*	*6.90%*	*11.58%*
	(-5.01% , -4.76%)	(0.11% , 0.27%)	(3.51% , 3.66%)	(6.82% , 6.98%)	(11.46% , 11.70%)

Table 13: Confidence intervals – single contribution (CIR-SVJD)

Regular contributions

Strategy	5%	25%	Median	75%	95%
Balanced A	*-0.24%*	*2.30%*	*4.07%*	*5.85%*	*8.43%*
	(-0.30% , -0.18%)	(2.26% , 2.34%)	(4.03% , 4.10%)	(5.81% , 5.89%)	(8.37% , 8.50%)
Life-cycle A	*0.08%*	*2.30%*	*3.96%*	*5.74%*	*8.51%*
	(0.03% , 0.13%)	(2.26% , 2.33%)	(3.93% , 4.00%)	(5.70% , 5.78%)	(8.43% , 8.58%)
Balanced B	*-5.00%*	*0.33%*	*4.07%*	*7.85%*	*13.39%*
	(-5.13% , -4.88%)	(0.25% , 0.42%)	(3.99% , 4.15%)	(7.76% , 7.93%)	(13.24% , 13.53%)
Life-cycle B	*-5.36%*	*0.28%*	*4.13%*	*7.94%*	*13.38%*
	(-5.49% , -5.22%)	(0.19% , 0.37%)	(4.04% , 4.21%)	(7.85% , 8.02%)	(13.24% , 13.53%)
Balanced C	*-4.26%*	*0.46%*	*3.76%*	*7.09%*	*11.97%*
	(-4.37% , -4.15%)	(0.38% , 0.53%)	(3.69% , 3.83%)	(7.02% , 7.16%)	(11.84% , 12.09%)
Life-cycle C	*-4.12%*	*0.47%*	*3.73%*	*7.06%*	*11.97%*
	(-4.23% , -4.01%)	(0.39% , 0.55%)	(3.66% , 3.81%)	(6.98% , 7.14%)	(11.85% , 12.09%)

Table 14: Confidence intervals – regular contributions (BS)

Strategy	5%	25%	Median	75%	95%
Balanced A	*-0.99%*	*2.07%*	*4.07%*	*6.00%*	*8.73%*
	(-1.07% , -0.91%)	(2.02% , 2.12%)	(4.02% , 4.11%)	(5.95% , 6.04%)	(8.66% , 8.80%)
Life-cycle A	*-0.82%*	*1.95%*	*3.94%*	*5.97%*	*9.06%*
	(-0.89% , -0.75%)	(1.90% , 2.00%)	(3.89% , 3.98%)	(5.93% , 6.02%)	(8.98% , 9.15%)
Balanced B	*-6.16%*	*0.14%*	*4.21%*	*8.10%*	*13.55%*
	(-6.33% , -5.99%)	(0.05% , 0.23%)	(4.12% , 4.29%)	(8.01% , 8.19%)	(13.40% , 13.69%)
Life-cycle B	*-6.70%*	*0.16%*	*4.34%*	*8.18%*	*13.31%*
	(-6.89% , -6.50%)	(0.06% , 0.27%)	(4.25% , 4.43%)	(8.09% , 8.27%)	(13.18% , 13.44%)
Balanced C	*-5.30%*	*0.27%*	*3.87%*	*7.32%*	*12.13%*
	(-5.45% , -5.16%)	(0.19% , 0.36%)	(3.79% , 3.95%)	(7.23% , 7.40%)	(12.00% , 12.26%)
Life-cycle C	*-5.29%*	*0.27%*	*3.85%*	*7.29%*	*12.09%*
	(-5.46% , -5.13%)	(0.18% , 0.35%)	(3.77% , 3.92%)	(7.20% , 7.37%)	(11.98% , 12.21%)

Table 15: Confidence intervals – regular contributions (CIR-SVJD)

D More life-cycle and corresponding balanced funds for regular contributions

Table 16 shows the matching balanced funds for more examples of different life-cycle funds when annual contributions are considered. We assume capital market parameters as given in Table 1, glide paths as indicated below and a management fee of 1.3% $p.a.$ The first three rows of Table 16 show strategies A, B, C, the next four give results for "wild" combinations of these strategies. Next, the following 12 rows treat strategies with only one year of complete investment in the risky asset whereas during the other years, the life-cycle funds considered are invested in the risk-free asset. Finally, the last 12 rows show results for the exact opposite.

Life-cycle fund‘s glide path given by their equity portion over time												Equity portion x_S	Management fee c_{BF}
1	0.9091	0.8182	0.7273	0.6364	0.5455	0.4545	0.3636	0.2727	0.1818	0.0909	0	35.88%	1.323%
0	0.0909	0.1818	0.2727	0.3636	0.4545	0.5455	0.6364	0.7273	0.8182	0.9091	1	76.08%	1.630%
1	0	1	0	1	0	1	0	1	0	1	0	67.25%	1.911%
1	0.9091	0.8182	0.7273	0.6364	0.5455	0.5455	0.6364	0.7273	0.8182	0.9091	1	79.70%	1.389%
0	0.0909	0.1818	0.2727	0.3636	0.4545	0.4545	0.3636	0.2727	0.1818	0.0909	0	26.27%	1.391%
1	0.0909	1	0.7273	0.3636	0	0.4545	0.6364	1	0.1818	0.9091	0	61.47%	1.665%
1	0.0909	0.8182	0	0.3636	0.5455	1	0.6364	0.2727	0	0.9091	0	56.54%	1.695%
1	0	0	0	0	0	0	0	0	0	0	0	4.63%	1.394%
0	1	0	0	0	0	0	0	0	0	0	0	9.08%	1.484%
0	0	1	0	0	0	0	0	0	0	0	0	13.36%	1.571%
0	0	0	1	0	0	0	0	0	0	0	0	17.46%	1.653%
0	0	0	0	1	0	0	0	0	0	0	0	21.40%	1.732%
0	0	0	0	0	1	0	0	0	0	0	0	25.17%	1.807%
0	0	0	0	0	0	1	0	0	0	0	0	28.79%	1.879%
0	0	0	0	0	0	0	1	0	0	0	0	32.25%	1.948%
0	0	0	0	0	0	0	0	1	0	0	0	35.58%	2.013%
0	0	0	0	0	0	0	0	0	1	0	0	38.76%	2.075%

0	0	0	0	0	0	0	0	0	0	1	0	41.81%	2.135%
0	0	0	0	0	0	0	0	0	0	0	1	44.72%	2.191%
0	1	1	1	1	1	1	1	1	1	1	1	99.78%	1.340%
1	0	1	1	1	1	1	1	1	1	1	1	99.33%	1.370%
1	1	0	1	1	1	1	1	1	1	1	1	98.71%	1.393%
1	1	1	0	1	1	1	1	1	1	1	1	97.97%	1.410%
1	1	1	1	0	1	1	1	1	1	1	1	97.16%	1.422%
1	1	1	1	1	0	1	1	1	1	1	1	96.29%	1.431%
1	1	1	1	1	1	0	1	1	1	1	1	95.40%	1.438%
1	1	1	1	1	1	1	0	1	1	1	1	94.51%	1.442%
1	1	1	1	1	1	1	1	0	1	1	1	93.63%	1.445%
1	1	1	1	1	1	1	1	1	0	1	1	92.79%	1.447%
1	1	1	1	1	1	1	1	1	1	0	1	91.98%	1.449%
1	1	1	1	1	1	1	1	1	1	1	0	91.21%	1.451%

Table 16: More numerical examples of life-cycle and their corresponding balanced funds

It is worthwhile noting that the sufficient conditions of Lemma 3 in Appendix A hold for all of these very different glide paths and hence a corresponding balanced fund can be derived. Further, we conclude the calibrated management fee $c_{BF} \geq 1.3\%\, p.a.$ for all life-cycle funds considered. Hence, a balanced fund equipped with above equity portion x_S and a management fee of exactly $1.3\%\, p.a.$ (equal to the life-cycle's management fee) may yield an expected return higher than when investing in the life-cycle fund and at the same time may provide a similar variability in returns.