Solvency II and Nested Simulations – a Least-Squares Monte Carlo Approach

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Motivation

- **Solvency II**: New regulatory framework for insurance companies in the European Union
- **Key aspect**: Determine required risk capital (SCR) for a one-year time horizon based on a market-consistent valuation of assets and liabilities
- **Standard model**: Approximation of SCR via square-root formula ⇒ Various deficiencies (cf. Pfeifer/Strassburger (2008), Sandström (2007)).
- **Alternative**: Multivariate approach based on stochastic model for the insurance company (Internal Model).
- **Problems**:
  - Valuation of life insurance contracts in closed form not possible (due to embedded options and guarantees)
  - Unsolved numerical and computational problems

⇒ This paper provides a mathematical framework for the calculation of the SCR and discusses different approaches for the numerical implementation.
Definitions

Assessment of solvency position can be split into two components:

1. Available Capital ($AC_0$)
   - Amount of financial resources available at time $t = 0$ which can serve as a buffer against risks and absorb financial losses.

Market consistent valuation of assets and liabilities

\[ AC_0 := MVA_0 - MVL_0 = MCEV_0 \]

- $MCEV_0$ denotes the market consistent embedded value, i.e. $MCEV_0 = ANAV_0 + PVFP_0 - CoC_0$, where
  - $ANAV_0$ is derived from statutory shareholders’ equity,
  - $PVFP_0$ is the present value of past-taxation shareholder cash flows from the assets backing (statutory) liabilities and
  - $CoC_0$ is the Cost-of-Capital charge (not discussed further here).

- Main computational issue: calculation of $PVFP_0$. 
Definitions

Assessment of solvency position can be split into two components:

2. Solvency Capital Requirement (SCR)
   - SCR is based on the Available Capital at $t = 1$, where $AC_1 := MCEV_1 + X_1$ and $X_1$ denotes shareholder cash flows at $t = 1$.
   - Intuition: An insurance company is considered solvent under Solvency II if its Available Capital at $t = 1$ is positive with a probability of at least $\alpha = 99.5\%$.
   - Therefore consider loss function $L := AC_0 - AC_1/(1 + i)$ where $i$ denotes the one-year risk-free rate at $t = 0$.

SCR definition

$$SCR := \arg\min_x \left\{ P \left( \frac{AC_0 - AC_1}{1 + i} > x \right) \leq 0.5\% \right\} = \text{VaR}_{99.5\%}(L)$$

- Main computational issue: calculation of 99.5%-quantile of $-AC_1$. 
Mathematical Framework

- Complete filtered probability space \((\Omega, \mathcal{F}, \mathcal{P}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]})\)
  - \(\mathcal{P}\) so-called real-world (physical) measure
  - Risk-neutral measure \(Q\) equivalent to \(\mathcal{P}\)
- State process: \((Y_t)_{t\in[0,T]} = \left(Y_{t}^{(1)}, \ldots, Y_{t}^{(d)}\right)_{t\in[0,T]}\) of sufficiently regular Markov processes that describes the stochasticity of the market
  - Numéraire process: \(B_t = \exp\left(\int_0^t r_u du\right), r_t = r(t, Y_t)\)
- Cash flow projection model, i.e. the future profits of the insurance company \(X_t (t = 1, \ldots, T)\) can be described as
  \[X_t = f_t(Y_s, s \in [0, t])\]
Valuation at $t = 0$

- Target: $AC_0 = ANAV_0 + \mathbb{E}^{Q} \left[ \sum_{t=1}^{T} \exp\left(-\int_{0}^{t} r_u \, du\right) X_t \right]$
  from statutory balance sheet
  $=: V_0$

- Problem: No closed form solution for $V_0$

- Monte Carlo simulations: $\tilde{V}_0(K_0) = \frac{1}{K_0} \sum_{k=1}^{K_0} \sum_{t=1}^{T} \exp\left(-\int_{0}^{t} r_u^{(k)} \, du\right) X_t^{(k)}$
Valuation at $t = 1$

Target: Distr. of $AC_1 = ANAV_1 + \mathbb{E}^Q \left[ \sum_{t=2}^{T} \exp(- \int_1^t r_u du) X_t \bigg| (Y_1, D_1) \right] + X_1 \sim F =: V_1$

Simulate $N$ first-year paths "under $P":\ (Y_1^{(i)}, D_1^{(i)})$
Simulate $K_1$ paths "under $Q"$ starting in $(Y_1^{(i)}, D_1^{(i)})$: determine $V_1^{(i)}$

$N \times K_1$ paths
Estimator in the Nested Simulations Approach

**Estimated SCR**

We now have

1. \( \widetilde{AC}_0(K_0) = \text{ANAV}_0 + \widetilde{V}_0(K_0) \)

2. \( \widetilde{AC}_1^{(i)}(K_1) := \text{ANAV}_1^{(i)} + \widetilde{V}_1^{(i)}(K_1) + X_1^{(i)}, 1 \leq i \leq N. \)

Hence, we can estimate SCR by

\[
\widetilde{SCR} = \widetilde{AC}_0 + \frac{\widetilde{z}(m)}{1 + i}
\]

where \( \widetilde{z}(m) \) is the \( m^{th} \) order statistic of \( -\widetilde{AC}_1^{(i)} \) and \( m = \lceil N \cdot 0.995 + 0.5 \rceil \).

- Within the estimation process, we have three sources of error:
  1. Estimation of \( AC_0 \) with only \( K_0 \) sample paths
  2. Estimation of the quantile with only \( N \) real-world scenarios
  3. Estimation of \( AC_1^{(i)} \) with only \( K_1 \) inner simulations \( \forall i \)

⇒ Analysis of the resulting error in our estimate \( \widetilde{SCR} \)
Variance-Bias Tradeoff – Choice of $K_0$, $N$ and $K_1$

▶ Idea: minimize the Mean-Square Error (MSE)

$$\text{MSE} = \mathbb{E} \left[ (\widehat{\text{SCR}} - \text{SCR})^2 \right] = \text{Var}(\widehat{\text{SCR}}) + \left[ \mathbb{E}(\widehat{\text{SCR}}) - \text{SCR} \right]^2 \text{bias}$$

▶ Similar to Gordy/Juneja (2008), we obtain:

**Optimization problem in $K_0$, $N$ and $K_1$**

$$\frac{\sigma_0^2}{K_0} + \frac{\alpha(1 - \alpha)}{(N + 2)f^2(\text{SCR})} + \frac{\theta_\alpha^2}{K_1^2 \cdot f^2(\text{SCR})} \rightarrow \min$$

subject to the effort restriction $K_0 + N \cdot K_1 = \Gamma$.

▶ Can be solved using Lagrangian multipliers (for given computational budget $\Gamma$).

▶ Note: bias is positive in practical applications resulting in a **systematic overestimation of the SCR**.

▶ Problem: To make bias small (for 99.5% confidence level), $K_1$ may not be chosen "too small" → **Immense computational effort!**
Least-Squares-Algorithm

- Based on LSM approach by Longstaff/Schwartz (2001) for the valuation of non-European options (see also Clément et al. (2002)).

**Algorithm:**
- Simulate $N$ scenarios (first year $\mathcal{P}$, other years $\mathcal{Q}$)

\[
PV_{1}^{(i)}(\omega_{i}) := \sum_{t=2}^{T} \exp \left\{ - \int_{1}^{t} r_{s}(\omega_{i}) \, ds \right\} \Psi_{t}(\omega_{i}) = \mathbb{E}^{Q} \left[ PV_{1}^{(i)} \mid \mathcal{F}_{1} \right] + \varepsilon_{i}, \quad 1 \leq i \leq N
\]

- $1^{st}$ step: Approximate $V_{1}$ by finite sum of appropriate basis functions

\[
V_{1} = \mathbb{E}^{Q} \left[ \sum_{t=2}^{T} \exp(- \int_{1}^{t} r_{u} \, du) \Psi_{t} \mid (Y_{1}, D_{1}) \right] \approx \hat{V}_{1}^{(M)}(Y_{1}, D_{1}) = \sum_{k=1}^{M} \alpha_{k} \cdot e_{k}(Y_{1}, D_{1})
\]

- $2^{nd}$ step: Estimate unknown parameter vector $\alpha$ via regression:

\[
\hat{\alpha}^{(N)} = \arg \min_{\alpha \in \mathbb{R}^{M}} \left\{ \sum_{i=1}^{N} \left[ PV_{1}^{(i)} - \sum_{k=1}^{M} \alpha_{k} \cdot e_{k} \left( Y_{1}^{(i)}, D_{1}^{(i)} \right) \right]^{2} \right\}
\]

- Estimate Available Capital:

\[
\hat{AC}_{1}^{(i)} = \text{ANAV}_{1}^{(i)} + \sum_{k=1}^{M} \hat{\alpha}_{k}^{(N)} \cdot e_{k}(Y_{1}^{(i)}, D_{1}^{(i)}) + \Psi_{1}^{(i)}(\omega_{i}), \quad 1 \leq i \leq N
\]
Least-Squares-Algorithm: Does it work?

Issues to consider:

► Suitability of regression approach
► Convergence of the algorithm
► Bias (finite number of basis functions, estimation of regression parameters)
► Choice of regression function

⇒ Ultimate test: How well does it perform in a somewhat realistic framework?
Example: A Participating Life Insurance Contract

- Term-fix insurance contract with minimum interest rate guarantee
- Bonus distribution models obligatory payments to the policyholder (MUST-case from Bauer et al. (2006))
- No mortality $\Rightarrow$ no biometric risk
- Dividends $d_t$ are paid to the shareholders
- Company obtains additional contribution $c_t$ from its shareholders in case of a shortfall
- Asset model: Extended Black-Scholes model with stochastic interest rates (see Bauer/Zaglauer (2008))
Bias in Nested Simulations, $N = 100,000$

- Choice of $K_1$ significantly affects SCR!
- Estimation of $\theta_\alpha$ via pilot simulation with $N = 100,000$, $K_1 = 100$ and regression/finite difference approximation:
  \[ \hat{\theta}_\alpha \approx 0.027 \Rightarrow (K_0; N; K_1) = (2,500,000; 550,000; 400) \text{ approx. optimal} \]
- Calculation takes about 35 minutes.
Comparison of different \((K_0; N; K_1)\) with \(\Gamma = 222,500,000\)

Based on 120 runs of simulations (approx. 35 min each)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>Mean (\widetilde{\text{SCR}})</th>
<th>Empirical Variance</th>
<th>Estimated Bias</th>
<th>Estimated MSE</th>
<th>Corrected Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>(275,000)</td>
<td>(800)</td>
<td>1319.6</td>
<td>28.0</td>
<td>1.5</td>
<td>30.2</td>
<td>1318.1</td>
</tr>
<tr>
<td>(550,000)</td>
<td>(400)</td>
<td>1320.5</td>
<td>19.3</td>
<td>3.0</td>
<td>28.2</td>
<td>1317.5</td>
</tr>
<tr>
<td>(1,100,000)</td>
<td>(200)</td>
<td>1323.1</td>
<td>8.8</td>
<td>5.9</td>
<td>43.9</td>
<td>1317.2</td>
</tr>
<tr>
<td>(2,200,000)</td>
<td>(100)</td>
<td>1328.9</td>
<td>4.4</td>
<td>11.8</td>
<td>143.2</td>
<td>1317.1</td>
</tr>
</tbody>
</table>

Table: Choice of \(N\) and \(K_1\) \((K_0 = 2,500,000)\), 120 runs
### Choice of the Regression Function in the LSM Approach

<table>
<thead>
<tr>
<th>#</th>
<th>Regression Function</th>
<th>Mean (SCR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1$</td>
<td>921.1</td>
</tr>
<tr>
<td>2</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2$</td>
<td>1141.9</td>
</tr>
<tr>
<td>3</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1$</td>
<td>1309.2</td>
</tr>
<tr>
<td>4</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2$</td>
<td>1330.1</td>
</tr>
<tr>
<td>5</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1$</td>
<td>1297.5</td>
</tr>
<tr>
<td>6</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1$</td>
<td>1302.5</td>
</tr>
<tr>
<td>7</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1 + \hat{\alpha}_7^{(N)} \cdot A_1 \cdot e^{r_1}$</td>
<td>1309.2</td>
</tr>
<tr>
<td>8</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1 + \hat{\alpha}_7^{(N)} \cdot A_1 \cdot e^{r_1}$ (+ \hat{\alpha}_8^{(N)} \cdot L_1 \cdot e^{r_1})</td>
<td>1316.5</td>
</tr>
<tr>
<td>9</td>
<td>$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1 + \hat{\alpha}_7^{(N)} \cdot A_1 \cdot e^{r_1}$ (+ \hat{\alpha}_8^{(N)} \cdot L_1 \cdot e^{r_1} + \hat{\alpha}_9^{(N)} \cdot e^{A_1/10000})</td>
<td>1317.5</td>
</tr>
</tbody>
</table>

**Table:** Estimated SCR for different choices of the regression function, $N = 550,000$

- Influence of basis function is quite pronounced.
- For "good" choices, the estimated SCR is close to the result obtained via Nested Simulations.
- "Good" choices appear to remain "good" for different parameters.
- Calculation takes only about 30 seconds.
Comparison of different $N$ in the LSM Approach

![Graph showing comparison of different N](image)

<table>
<thead>
<tr>
<th>$N$</th>
<th>Mean ($\hat{SCR}$)</th>
<th>Empirical Variance</th>
<th>Solvency Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>275,000</td>
<td>1316.9</td>
<td>87.5</td>
<td>132%</td>
</tr>
<tr>
<td>550,000</td>
<td>1317.5</td>
<td>62.6</td>
<td>132%</td>
</tr>
<tr>
<td>1,100,000</td>
<td>1317.4</td>
<td>23.5</td>
<td>132%</td>
</tr>
<tr>
<td>2,200,000</td>
<td>1317.2</td>
<td>10.5</td>
<td>132%</td>
</tr>
</tbody>
</table>

Table: Results for the LSM estimator, 120 runs
Summary

▶ Nested Simulations:
  → Inadequate choice of \((K_0, N, K_1)\) in nested simulations may yield erroneous outcomes.
  → Immense computational effort to achieve accurate results.

▶ LSM:
  → Fast approach to achieve relatively accurate results.
  → Results are similarly positive when calculating SCR for longer time horizons ("richer sigma field").
  → Care is required in choice of regression function even though simple algorithms yield good results in our applications.
  → Open question: theoretical results regarding validity of approximation.

Future Research

▶ Improvement of the Nested Simulations Approach by variance reduction techniques, QMC and screening procedures.
▶ Use of statistical methods to determine the regression function.
▶ Analysis of other risk measures, such as TVaR.
Literature


Thanks for your attention!