THE PRICING OF IMPLICIT OPTIONS IN LIFE INSURANCE CONTRACTS A GENERAL APPROACH USING MULTIVARIATE TREE STRUCTURES

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ABSTRACT

In the present paper we introduce a new model for the pricing of implicit options in life insurance contracts. Many models used so far allow only for one source of uncertainty. Since many options – in particular within unit-linked contracts – depend on the stochastic behavior of both interest rates and the underlying assets, we develop a model that includes both risks. Since implicit options in life insurance contracts are often American or Bermuda-style options, we introduce a multivariate tree structure, where the asset price risk is modeled by a binomial tree according to the Cox-Ross-Rubinstein model, and interest rates are modeled by a trinomial tree according to the Hull-White model. We give some possible applications of the model and explain how our previous work is included in the model as a special case.

1. INTRODUCTION

An implicit option is the right to change some product features or to choose between some alternatives at some time during the term of a life insurance policy. Some well known examples are the so-called guaranteed insurability options, where the insured person has the right to increase the death benefit at the occurrence of certain events (like marriage, birth of a child, etc), the lump sum option in deferred annuities, where the insured person can choose between a lump sum and a lifelong annuity (see [Di/Ru 99] for a detailed analysis of this option), or the flexible expiration option, where the insured can terminate his contract at any time during a predefined interval (see [Di/Ru 00] for a detailed analysis of this option).

Such options can have an extreme impact on the cash flow that results from the policy. Hence, they can bear significant financial risks that are often not taken into account when the policy is priced. In spite of these risks, options in life insurance contracts are becoming more and more popular since they make the product more competitive. Furthermore, in some countries, e.g. Germany, changing a contract by an option that was given at the beginning is sometimes preferable to a 'real' change of the contract because of tax reasons.

Some of these options can be regarded as financial options – either derivatives on the underling assets or on interest rates. The right to 'sell' a lifelong annuity and receive a lump sum instead is e.g. essentially equivalent to a European put option on a coupon bond.[1] Such interest rate sensitive options are of particular interest in the German life insurance market, since most German life insurance companies try to give their policy-holders the same return every year, independent of what they earn on their investment. This smoothening is achieved by accumulating hidden reserves when the markets perform well, and using these reserves to distribute the same profits every year, even if the markets perform bad. Hence, there are times when the interest earned on life insurance contracts is significantly higher than the rates earned in the bond markets, and vice versa. Policy-holders might use implicit options to profit from these effects. Thus, a detailed analysis of such options and the risk they impose on the insurance company is required.

Since market shares of unit-linked contracts have risen dramatically over the last years, analyzing the corresponding options in unit-linked contracts is necessary, too. However, the situation is more complicated here, since we not only need to model the interest rate behavior but also the price of the underlying asset.

Our paper is organized as follows: In Section 2 and 3, we introduce the Cox-Ross-Rubinstein model and Hull-White model, respectively. . In Section 4 we introduce our new multivariate tree model and give some possible applications in Section 5. Section 6 closes with a short summary and an outlook for further research. In Section 7 and 8, we list the references and give some details on the computations done within our multivariate tree model, respectively.

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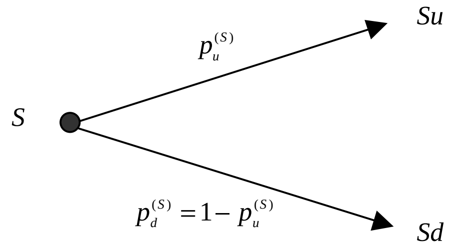
[1] Cf. [Di/Ru 99] for details.

2. THE COX-ROSS-RUBINSTEIN MODEL

In [Co/Ro/Ru 79], Cox, Ross, and Rubinstein introduced their discrete-time model for the pricing of options. It is particularly popular when dealing with non-European style options because of its simplicity. The stock price is modeled by a binomial tree. In this section, we will give a brief description of this model and how the so-called principle of risk-neutral-valuation is applied within the model.[2]

Let us consider a non-dividend-paying stock. Suppose our observation starts at time $t = 0$ and we consider time-steps of length $\Delta t > 0$. Furthermore, we let *S* denote the price of the stock at the beginning. In each of the small time intervals, the stock price can either move up or down, that is, it can change to *Su* or *Sd*, with $u > 1$ and $d < 1$. In the risk-neutral world, the probability for moving up will be denoted by $p_u^{(S)}$. Thus, the probability of a down movement is $p_d^{(S)} = 1 - p_u^{(S)}$ (cf. Figure 1).

Figure 1: Stock price movement in time Δt under the binomial model



Thus, for every $t > 0$ we have

$$\frac{\Delta S}{S} = \frac{S_{t+\Delta t} - S_t}{S_t} = \begin{cases} u - 1 & \text{with probability } p_u^{(S)} \\ d - 1 & \text{with probability } p_d^{(S)} \end{cases}.$$

The parameters *u*, *d*, and $p_u^{(S)}$ are determined such that both the mean and the variance of the stock price during a time interval of length Δt in the binomial tree match their theoretical values. This is where the principle of risk-neutral-valuation comes into play. It means that any security dependent on the stock price can be valued on the assumption that the world is risk-neutral. That is, for the purpose of valuing an option, we can assume that the expected return from all traded securities is the risk-free interest rate. Furthermore, future cash flows can be valued by discounting their expected values at the risk-free interest rate which we will denote by *r*.

As a result, we get

$$Se^{r\Delta t} = p_u^{(S)} Su + (1 - p_u^{(S)}) Sd$$

and

$$S^2 e^{2r\Delta t}(e^{\sigma^2 \Delta t} - 1) = p_u^{(S)} S^2 u^2 + (1 - p_u^{(S)}) S^2 d^2 - S^2 \left(p_u^{(S)} u + (1 - p_u^{(S)}) d\right)^2,$$

where σ denotes the stock price volatility. These equations can be simplified to

$$e^{r\Delta t} = p_u^{(S)} u + (1 - p_u^{(S)}) d \tag{1}$$

and

$$e^{2r\Delta t + \sigma^2 \Delta t} = p_u^{(S)} u^2 + (1 - p_u^{(S)}) d^2 . \tag{2}$$

As a third equation, Cox, Ross, and Rubinstein propose

$$u = d^{-1} . \tag{3}$$

Equations (1), (2), and (3) imply (cf. [Hu 97])

$$p_u^{(S)} = \frac{e^{r\Delta t} - d}{u - d}$$

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}} .$$ [3]

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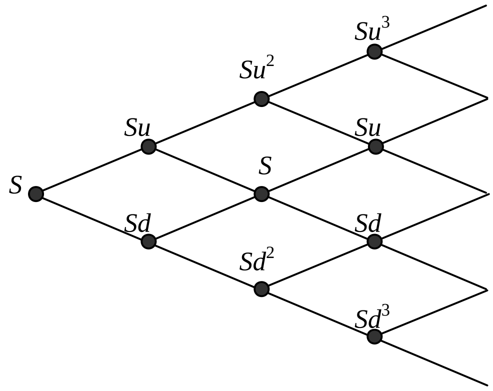
[2] Cf. e.g. [Du 96] or [Mu/Ru 97].

[3] Here, terms of higher order than Δt are ignored. The exact solutions can e.g. be found in [Hu 97].

Because of equation (3), the tree is recombining (i.e. an up movement followed by a down movement leads to the same stock price as a down movement followed by an up movement). In general, the node (i, j) corresponds to the time $i\Delta t$ with a stock price of $Su^j d^{i-j}$ with $j = 0,1,...,i$. Note that (3) is used in computing the stock price at each node of the tree in Figure 2 (e.g. $Su^2 d = Su$).

Figure 2: Binomial tree of the stock price



The valuation of an option on this stock is then done by the so-called method of backward induction, beginning at the time of maturity *T* of the option. The value of the option at *T* is known. For example, a put option on the stock is worth $\max[X - S_T ;0]$, where S_T is the stock price at time *T* and *X* is the strike price of the option. In a risk-neutral world, the value of the option at some time $0 < i\Delta t < T$ can be computed as the expected value (using the risk-neutral probability $p_u^{(S)}$) of the option at time $(i+1)\Delta t$ discounted for a period Δt at the risk-free rate *r*. If it is not a European-style option, it is necessary to check at each node whether early exercise is preferable to holding the option for a further time period Δt . By working backwards through the tree like this, the value of the option at time zero is eventually obtained.

The same method can be used to price options on dividend-paying stocks, indices, currencies, and futures contracts (see [Hu 97] for details).

Cox, Ross, and Rubinstein also showed in [Co/Ro/Ru 79] that a special limiting case of their model coincides with the Black-Scholes model for option pricing, where the stock price is given by the process

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t) \text{ .}$$ [4]

3. THE HULL-WHITE MODEL

In [Hu/Wh 90], Hull and White introduced their model for the term structure. In this section, we give a short introduction to this one-factor, no-arbitrage model for the short rate which is assumed to follow the Itô-process

$$\begin{aligned} dr(t) &= (\theta(t) - ar(t))dt + \sigma\, dW(t) \\ &= a\left(\frac{\theta(t)}{a} - r(t)\right)dt + \sigma\, dW(t), \end{aligned}$$

with constant $a, \sigma > 0$.

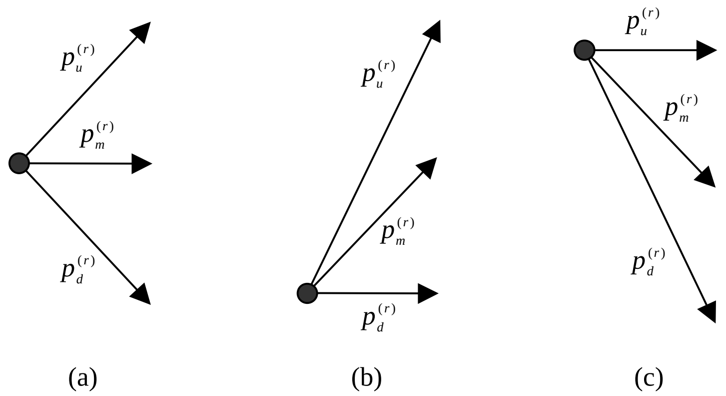
This model includes mean reversion, that is, at time *t*, the short rate $r(t)$ reverts to $\theta(t)/a$ with mean reversion rate *a*. Hence, for large values of $r(t)$, there is a negative drift of $\theta(t) - ar(t)$ that pulls the short rate back to the time dependent drift term $\theta(t)/a$. Similarly, for small values of $r(t)$, the short rate is pulled up by the drift.

Since the model of Hull and White is a no-arbitrage model, it can be fitted to any given term structure by adjusting $\theta(t)$.

[4] Cf. [Bl/Sc 73].

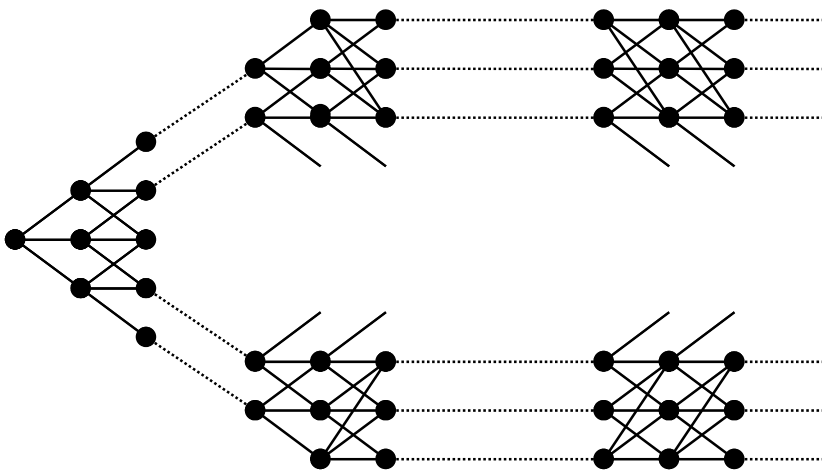
A positive aspect of the Hull-White model is that – even when there exist no closed form solutions (as is the case with American or Bermuda-style options) – it can easily be implemented. In [Hu/Wh 94], Hull and White propose the use of a trinomial tree. Its advantage over a binomial tree is that it offers one more degree of freedom that is helpful in realizing mean reversion.

Figure 3: Branching in trinomial tree



Usually, the tree branches as in Figure 3(a). If the short rate is small, the tree will branch off as in Figure 3(b). That reflects the mean reversion which implies a stronger upward drift. In case the short rate is rather large, the branching pattern as shown in Figure 3(c) will be used. Therefore, the tree will have a structure as shown in Figure 4.

Figure 4: Trinomial tree of the short rate



In [Hu/Wh 94], the authors give a criterion for the step length Δr of the short rate in the tree (depending on the length of the time steps Δt). They also give a criterion that specifies the point in time when the branching patterns given in Figure 3(b) and 3(c) will be used first and the tree will stop growing vertically (cf. Figure 4). This not only reflects the mean reversion of the model but also has the advantage that the dimension of the tree is much smaller than that of a binomial tree.

We will not go into all the details of constructing and calibrating this tree but only give an idea about the different steps of this procedure. For the complete technical details we refer the reader to [Hu/Wh 94].

In a first step, the tree is constructed for the simple process $dr^*(t) = -ar^*(t)dt + \sigma dW(t)$. This process is symmetric around $r^* = 0$ and its increments are normally distributed. In a second step, the difference process $\alpha(t) = r(t) - r^*(t)$, that is, $d\alpha(t) = (\theta(t) - a\alpha(t))dt$, is used to compute the values of $r(t)$. This means that in every node we compute the interest rate for the next Δt period in the tree. However, this is not the instantaneous short rate r, but it can of course be computed from it (cf. [Hu 97]). Together with the explicit formula of bond prices within the model, we are able to compute the prices of zero bonds (and thus the entire term structure) in every node of the tree.

In order to compute the values of the short rate, the transitions probabilities in each node (i, j) to all its possible succeeding nodes need to be specified first. They not only depend on the position of the node within the tree, but also on the model parameters θ and σ. According to [Hu/Wh 94], they are chosen such that both the expected change and the expected variance of the short rate for the next time step Δt in the tree fit the corresponding values under the risk-neutral measure. As a third condition, the sum of all three transition probabilities has to be one. In the case of the standard branching shown in Figure 3(a), the probabilities are given by (cf. [Hu /Wh 94])

$$p_u^{(r)} = \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - ajt}{2}$$

$$p_m^{(r)} = \frac{2}{3} - a^2 j^2 (\Delta t)^2$$

$$p_d^{(r)} = \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 + ajt}{2} .$$

Similarly, if the branching has the form shown in Figure 3(b), the probabilities are

$$p_u^{(r)} = \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 + ajt}{2}$$

$$p_m^{(r)} = -\frac{1}{3} - a^2 j^2 (\Delta t)^2 - 2aj\Delta t$$

$$p_d^{(r)} = \frac{7}{6} + \frac{a^2 j^2 (\Delta t)^2 + 3ajt}{2} .$$

Finally, for the branching of Figure 3(c), the probabilities are

$$p_u^{(r)} = \frac{7}{6} + \frac{a^2 j^2 (\Delta t)^2 - 3ajt}{2}$$

$$p_m^{(r)} = -\frac{1}{3} - a^2 j^2 (\Delta t)^2 + 2aj\Delta t$$

$$p_d^{(r)} = \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - ajt}{2} .$$

So if we let $i^* \Delta t$ denote the point in time where the interest rate tree uses the non-standard branching pattern shown in Figure 3(b) and 3(c) for the first time, then for $i \in \{0,1,2,\ldots\}$ and $j \in \{-\tilde{i}, -\tilde{i}+1, \ldots, 0, \ldots, \tilde{i}-1, \tilde{i}\}$ with $\tilde{i} = \min[i; i^*]$, the node (i, j) in the tree corresponds to time $i\Delta t$ with a short rate of $j\Delta r + \alpha(i\Delta t)$.[5]

The valuation of any security using this tree is analogous to the valuation using backward induction in a binomial tree as described in the previous section.

4. TRIBINOMIAL MODEL

In order to analyze implicit options in unit-linked contracts we need to combine the model for the interest rate with the model for the stock price. Thus, we have

$$dS(t) = S(t)r(t)dt + S(t)\sigma_S dW_1(t) \text{ and}$$
$$dr(t) = (\theta(t) - ar(t))dt + \sigma_r dW_2(t), \tag{4}$$

with two (in general correlated) Wiener processes $W_1(t)$ and $W_2(t)$.

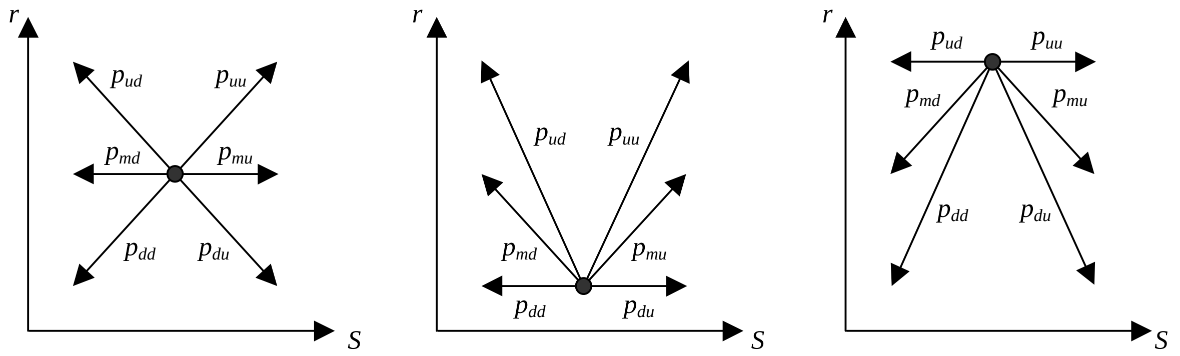
Since we want to be able to price options that can be exercised not only at a single point in time, it seems to be natural to combine the two tree structures described above into one single tree, allowing for correlation. We will call this combined tree a tribinomial tree, which is a discrete version of (4).

As a consequence, at each node in the new tree there are six possible succeeding nodes. In the standard case, they correspond to a combination of a movement of the stock (up or down) and a movement of the interest rate (up, down, or no change), as illustrated in Figure 5 (a).

[5] The exact condition for i^* can be found in [Hu/Wh 94].

Similar to the trinomial tree of the short rate, we have a different branching pattern when interest rates are very low, i.e. the tree branches according to Figure 5 (b), or very high, i.e. the tree branches according to Figure 5 (c).

Figure 5: Branching in tribinomial tree



(a) (b) (c)

Since this tree has three dimensions, each node can be characterized by some triple (i, j, k), with $i \in \{0,1,2,\ldots\}$, $j \in \{-\tilde{i}, -\tilde{i}+1, \ldots, 0, \ldots, \tilde{i}-1, \tilde{i}\}$, and $k \in \{0,1,2,\ldots,i\}$, with $\tilde{i} = \min[i; i^*]$ as introduced in the previous section. Here, the node (i, j, k) corresponds to time $i\Delta t$ with a short rate of $j\Delta r + \alpha(i\Delta t)$ and a stock price of $Su^k d^{k-i}$ (cf. Sections 2 and 3).

Furthermore, we will denote the transition probabilities in node (i, j, k) by $p_{\mu\nu}(i, j, k)$, with $\mu \in \{u, m, d\}$ and $\nu \in \{u, d\}$. In order for the tribinomial tree too be consistent with both a given trinomial tree of the short rate and a given binomial tree of the stock price, these probabilities have to satisfy all of the following equations:

$$p_{uu}(i, j, k) + p_{ud}(i, j, k) = p_u^{(r)}(i, j) \quad (5)$$

$$p_{mu}(i, j, k) + p_{md}(i, j, k) = p_m^{(r)}(i, j) \quad (6)$$

$$p_{du}(i, j, k) + p_{dd}(i, j, k) = p_d^{(r)}(i, j) \quad (7)$$

$$p_{ud}(i, j, k) + p_{md}(i, j, k) + p_{dd}(i, j, k) = p_d^{(S)}(i, k) \quad (8)$$

$$p_{uu}(i, j, k) + p_{mu}(i, j, k) + p_{du}(i, j, k) = p_u^{(S)}(i, k) \quad (9)$$

Here, $p_\mu^{(r)}(i, j)$ and $p_\nu^{(S)}(i, k)$ refer to the probabilities of the trinomial tree for the short rate and the binomial tree for the stock price, respectively (cf. Sections 2 and 3).

Since we also have $p_u^{(r)}(i, j) + p_m^{(r)}(i, j) + p_d^{(r)}(i, j) = 1$ and $p_u^{(S)}(i, k) + p_d^{(S)}(i, k) = 1$, only four of the equations above are linearly independent. Thus, we will use (5)-(8) and ignore (9) (which is satisfied implicitly).

In order to simplify the notation, we will drop the index (i, j, k) of the probabilities. Furthermore, we will use $R = R(i, j)$ as the random variable of the short rate at node (i, j, k) with $R(i, j) = \tilde{r}$. Similarly, we will use $S = S(i, k)$ as the random variable of the stock price at node (i, j, k) with $S(i, k) = \tilde{s}$. The relative change in the stock price over a period of Δt will be denoted by ΔS/S. Note, that we have

$$\frac{\Delta S}{S} = \frac{S(i+1, l) - S(i, k)}{S(i, k)} = \begin{cases} \dfrac{u\tilde{s} - \tilde{s}}{\tilde{s}} = u - 1 & \text{for } l = k+1, \text{and thus } S(i+1) = u\tilde{s} \\ \dfrac{d\tilde{s} - \tilde{s}}{\tilde{s}} = u - 1 & \text{for } l = k, \text{and thus } S(i+1) = d\tilde{s} \end{cases}.$$

In order to specify the model, we need two more equations. One further condition is given by the correlation between the interest rate process and the relative change in the stock price.[6] If $\hat{\rho}$ denotes a given estimate of this correlation (e.g. from market data), we use

[6] Since the stock price process has a drift and the interest rate process follows a mean reversion, it makes no sense to model the correlation between *r* and *S*. A frequent assumption in existing literature is, however, that the return of the stock (ΔS/S) and the short rate have a constant correlation (usually assumed to be negative).

$$\rho\left[R,\frac{\Delta S}{S}\right]=\hat{\rho}\,. \qquad (10)$$

Thus, depending on the three cases of Figure 5, we have[7]

$$\begin{aligned}(u-1)(p_{uu}-p_{du})+(d-1)(p_{ud}-p_{dd})&=(p_u^{(r)}-p_d^{(r)})\left((u-1)p_u^{(S)}+(d-1)p_d^{(S)}\right)\\&+\hat{\rho}\sqrt{\left(p_u^{(r)}+p_d^{(r)}-(p_u^{(r)}-p_d^{(r)})^2\right)\left(u^2p_u^{(S)}-d^2p_d^{(S)}-(up_u^{(S)}+dp_d^{(S)})^2\right)}\end{aligned} \qquad (10a)$$

$$\begin{aligned}(u-1)(p_{mu}+2p_{du})+(d-1)(p_{md}+2p_{dd})&=(p_m^{(r)}+2p_d^{(r)})\left((u-1)p_u^{(S)}+(d-1)p_d^{(S)}\right)\\&-\hat{\rho}\sqrt{\left(p_m^{(r)}+4p_d^{(r)}-(p_m^{(r)}+2p_d^{(r)})^2\right)\left(u^2p_u^{(S)}-d^2p_d^{(S)}-(up_u^{(S)}+dp_d^{(S)})^2\right)}\end{aligned} \qquad (10b)$$

$$\begin{aligned}(u-1)(2p_{uu}+p_{mu})+(d-1)(2p_{ud}+p_{md})&=(2p_u^{(r)}+p_m^{(r)})\left((u-1)p_u^{(S)}+(d-1)p_d^{(S)}\right)\\&+\hat{\rho}\sqrt{\left(4p_u^{(r)}+p_m^{(r)}-(2p_u^{(r)}+p_m^{(r)})^2\right)\left(u^2p_u^{(S)}-d^2p_d^{(S)}-(up_u^{(S)}+dp_d^{(S)})^2\right)}\end{aligned} \qquad (10c)$$

In order to find unique solutions for the six probabilities, we need one more equation. Therefore, we introduce the condition that the second moment of the interest rate in each node (i,j,k) is the same, independent of the behavior of the stock price. That is, the expected value of R^2 should be the same for both an up and a down movement of the stock price over the next Δt time period, i.e.

$$E\left[R^2\mid\frac{\Delta S}{S}=u-1\right]=E\left[R^2\mid\frac{\Delta S}{S}=d-1\right].^{8} \qquad (11)$$

Again, we get a different equation for each of the three cases of Figure 5:

$$(\tilde{r}+\Delta r)^2(p_{uu}-p_{ud})+\tilde{r}^2(p_{mu}-p_{md})+(\tilde{r}-\Delta r)^2(p_{du}-p_{dd})=0 \qquad (11a)$$

$$\tilde{r}^2(p_{uu}-p_{ud})+(\tilde{r}-\Delta r)^2(p_{mu}-p_{md})+(\tilde{r}-2\Delta r)^2(p_{du}-p_{dd})=0 \qquad (11b)$$

$$(\tilde{r}+2\Delta r)^2(p_{uu}-p_{ud})+(\tilde{r}+\Delta r)^2(p_{mu}-p_{md})+\tilde{r}^2(p_{du}-p_{dd})=0 \qquad (11c)$$

Therefore, in any of the three cases from Figure 5, we need to solve the equations (5), (6), (7), and (8), together with (10a), (10b), (10c), and (11a), (11b), (11c), respectively. The results in the case of Figure 5(a) are:

$$\begin{aligned}p_{uu}^{(a)}&=p_u^{(r)}p_u^{(S)}-(2\tilde{r}-\Delta r)A_1^{(a)}-A_2^{(a)}\\p_{mu}^{(a)}&=p_m^{(r)}p_u^{(S)}+4\tilde{r}A_1^{(a)}+2A_2^{(a)}\\p_{du}^{(a)}&=p_d^{(r)}p_u^{(S)}-(2\tilde{r}+\Delta r)A_1^{(a)}-A_2^{(a)}\\p_{ud}^{(a)}&=p_u^{(r)}p_d^{(S)}+(2\tilde{r}-\Delta r)A_1^{(a)}+A_2^{(a)}\\p_{md}^{(a)}&=p_m^{(r)}p_d^{(S)}-4\tilde{r}A_1^{(a)}-2A_2^{(a)}\\p_{dd}^{(a)}&=p_d^{(r)}p_d^{(S)}+(2\tilde{r}+\Delta r)A_1^{(a)}+A_2^{(a)}\end{aligned}$$

with

$$A_1^{(a)}=\frac{\hat{\rho}\sqrt{\sigma_{(a)}^2(R)\sigma^2\left(\frac{\Delta S}{S}\right)}}{2(u-d)(\Delta r)^2}\ \text{and}$$

$$A_2^{(a)}=\frac{\mu_{(a)}(R^2)}{4(\Delta r)^2}\left(p_u^{(S)}-p_d^{(S)}\right).$$

In the case of Figure 5(b):

$$\begin{aligned}p_{uu}^{(b)}&=p_u^{(r)}p_u^{(S)}-(2\tilde{r}-3\Delta r)A_1^{(b)}-A_2^{(b)}\\p_{mu}^{(b)}&=p_m^{(r)}p_u^{(S)}+(4\tilde{r}-4\Delta r)A_1^{(b)}+2A_2^{(b)}\end{aligned}$$

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[7] The computations can be found in Appendix 8.1.

[8] This condition, of course, is rather arbitrary, as is e.g. the condition $u=d^{-1}$ in the Cox-Ross-Rubinstein model. The idea is, that we assume the volatility of the interest rate process to be uninfluenced by movements of the stock price. In the model, however, we prefer the second moment over the variance, since in this case, the equations (11a), (11b), and (11c) are linear in the probabilities p_{uu}, p_{mu}, p_{du}, p_{ud}, p_{md}, and p_{dd}.

$$
\begin{aligned}
p_{du}^{(b)} &= p_d^{(r)} p_u^{(S)} - (2\tilde{r} - \Delta r) A_1^{(b)} - A_2^{(b)} \\
p_{ud}^{(b)} &= p_u^{(r)} p_d^{(S)} + (2\tilde{r} - 3\Delta r) A_1^{(b)} + A_2^{(b)} \\
p_{md}^{(b)} &= p_m^{(r)} p_d^{(S)} - (4\tilde{r} - 4\Delta r) A_1^{(b)} - 2A_2^{(b)} \\
p_{dd}^{(b)} &= p_d^{(r)} p_d^{(S)} + (2\tilde{r} - \Delta r) A_1^{(b)} + A_2^{(b)}
\end{aligned}
$$

with

$$
A_1^{(b)} = \frac{\hat{\rho}\sqrt{\sigma_{(b)}^2(R)\sigma^2\left(\frac{\Delta S}{S}\right)}}{2(u-d)(\Delta r)^2}
$$

and

$$
A_2^{(b)} = \frac{\mu_{(b)}(R^2)}{4(\Delta r)^2}\left(p_u^{(S)} - p_d^{(S)}\right).
$$

Finally, in the case of Figure 5(c):

$$
\begin{aligned}
p_{uu}^{(c)} &= p_u^{(r)} p_u^{(S)} - (2\tilde{r} + \Delta r) A_1^{(c)} - A_2^{(c)} \\
p_{mu}^{(c)} &= p_m^{(r)} p_u^{(S)} + (4\tilde{r} + 4\Delta r) A_1^{(c)} + 2A_2^{(c)} \\
p_{du}^{(c)} &= p_d^{(r)} p_u^{(S)} - (2\tilde{r} + 3\Delta r) A_1^{(c)} - A_2^{(c)} \\
p_{ud}^{(c)} &= p_u^{(r)} p_d^{(S)} + (2\tilde{r} + \Delta r) A_1^{(c)} + A_2^{(c)} \\
p_{md}^{(c)} &= p_m^{(r)} p_d^{(S)} - (4\tilde{r} + 4\Delta r) A_1^{(c)} - 2A_2^{(c)} \\
p_{dd}^{(c)} &= p_d^{(r)} p_d^{(S)} + (2\tilde{r} + 3\Delta r) A_1^{(c)} + A_2^{(c)}
\end{aligned}
$$

with

$$
A_1^{(c)} = \frac{\hat{\rho}\sqrt{\sigma_{(c)}^2(R)\sigma^2\left(\frac{\Delta S}{S}\right)}}{2(u-d)(\Delta r)^2}
$$

and

$$
A_2^{(c)} = \frac{\mu_{(c)}(R^2)}{4(\Delta r)^2}\left(p_u^{(S)} - p_d^{(S)}\right).
$$

For all cases, $\mu_{(\cdot)}(R^2)$ and $\sigma_{(\cdot)}^2(R)$ denote the expected value of R^2 and the variance of R, respectively, in the case of branching according to Figure 5(·). Furthermore, $\sigma^2\left(\Delta S/S\right)$ denotes the variance of $\Delta S/S$.[9]

With these formulas, it is rather simple to implement a general tree model for the pricing of derivatives that depend on both interest rates and some asset price.

5. APPLICATIONS

The model allows a variety of applications in the area of implicit options in life insurance contracts. First of all, for contracts where the assets of the insured person develop in a deterministic way (as e.g. can be assumed for non-linked contracts in Germany), we let the stock price volatility σ_S be zero and the model coincides with the model used in [Di/Ru 99] and [Di/Ru 00].

More interesting applications can be found in the area of single-premium unit-linked contracts. Consider for instance the lump sum option in deferred single premium unit-linked annuities. In this kind of contract, the insured invests a single premium into a fund. After the deferment period he has the right to either receive a lifelong, non-linked annuity (often calculated with a rate of interest already guaranteed at the beginning of the deferment period) or a lump sum. Of course, this lump sum option can be considered a put option on a coupon bond[10] exercisable at the end of the deferment period. Both, the amount of the annuity that is 'sold' when exercising the put and the lump sum, i.e. the exercise price that is received when exercising the put, depend heavily on the performance of the fund during the deferment period. Furthermore, the value of the bond depends

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[9] Their exact formulas are given in Appendix 8.2.
[10] The coupon bond represents the lifelong annuity.

on the term structure at the end of the deferment period. Hence, this option can only be valued within a combined model for the interest rate and the asset process.

Whilst the option described above could also be evaluated by a Monte-Carlo approach in a continuous model (cf. (4)), this is not possible, if the insured has the right to either choose the lump sum or the lifelong annuity at any time during a certain interval (e.g. between age 55 and 65). In this case, the option becomes a Bermuda-style option. However, it can still easily be analyzed within our model.

Furthermore, the model can be applied when pricing the value of guarantees in guaranteed unit-linked contracts. There exists a variety of literature on this topic.[11] However, if the contract specifies a guarantee at different dates, our model seems to be more feasible than the models considered so far. It could e.g. be used for the pricing of single premium guaranteed unit-linked contracts with flexible expiration. Here, the insured has the right to choose between the net asset value of some underlying fund investment and a deterministic lump sum, e.g. the payback of his single premium (perhaps including some guaranteed rate of interest), at several times (e.g. annually between age 55 and 65).

6. SUMMARY AND OUTLOOK

In the present paper we have introduced a multivariate tree model for the pricing of derivatives depending on some underlying asset and the term structure of interest rates. The model is rather easy to implement and allows for a variety of applications in the area of implicit options in life insurance contracts, in particular American or Bermuda-style options in single-premium unit-linked contracts.

A drawback of the model is the fact, that it can not handle regular premium contracts since in that case the binomial tree will not be recombining any more.[12]

The model includes all the properties of both, the Black/Scholes (or Cox/Ross/Rubinstein) and the Hull/White-model. Next steps in research will be the implementation and calibration of the model and the pricing of several options. It will be interesting to analyze the value and the sensitivity with respect to input parameters like volatilities and interest rate levels of such options that are often included for free in life insurance contracts.

A challenge in the implementation of the model will be the fact, that the model obviously requires the same step length for both trees. Thus, optimal step lengths for the individual processes (as given for the interest rate tree in [Hu/Wh 94]) will in general differ. Hence, finding a suitable step length for the combined tree might require some new ideas.

7. REFERENCES

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[12] If the insured person invests 100 units of currency every month, the value of his asset will be lower if the asset price process performs an up movement followed by a down movement than if the asset process moves first down and then up, since in the latter case more units or shares can be bought with the second premium payment.

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8. APPENDIX

8.1 Correlation between interest rate and relative change in stock price

The correlation between the interest rate R and the relative change in the stock price $\frac{\Delta S}{S}$ can be computed by

$$\rho\left[R,\frac{\Delta S}{S}\right]=\frac{Cov\left[R,\frac{\Delta S}{S}\right]}{\sqrt{Var[R]}\sqrt{Var\left[\frac{\Delta S}{S}\right]}}.$$

Furthermore, we know that

$$Cov\left[R,\frac{\Delta S}{S}\right]=E\left[R\frac{\Delta S}{S}\right]-E[R]E\left[\frac{\Delta S}{S}\right].$$

The expected value and the variance of the relative change in the stock price over the next time interval Δt is

$$\begin{aligned}E\left[\frac{\Delta S}{S}\right]&=(u-1)p_u^{(S)}+(d-1)p_d^{(S)}\\&=up_u^{(S)}+dp_d^{(S)}-1\end{aligned}$$

and

$$\begin{aligned}Var\left[\frac{\Delta S}{S}\right]&=E\left[\left(\frac{\Delta S}{S}\right)^2\right]-E\left[\frac{\Delta S}{S}\right]^2\\&=(u-1)^2p_u^{(S)}+(d-1)^2p_d^{(S)}-\left((u-1)p_u^{(S)}+(d-1)p_d^{(S)}\right)^2\\&=u^2p_u^{(S)}+d^2p_d^{(S)}-\left(up_u^{(S)}+dp_d^{(S)}\right)^2.\end{aligned}$$

For the expected value and the variance of the interest rate R over the next time interval Δt we have to distinguish the three different cases of Figure 5. In case of Figure 5(a), we have

$$\begin{aligned}E[R]&=(\tilde r+\Delta r)p_u^{(r)}+\tilde r p_m^{(r)}+(\tilde r-\Delta r)p_d^{(r)}\\&=\tilde r+\Delta r\left(p_u^{(r)}-p_d^{(r)}\right)\end{aligned}$$

and

$$\begin{aligned}Var[R]&=E[R^2]-E[R]^2\\&=(\tilde r+\Delta r)^2p_u^{(r)}+\tilde r^2p_m^{(r)}+(\tilde r-\Delta r)^2p_d^{(r)}-\left(\tilde r+\Delta r\left(p_u^{(r)}-p_d^{(r)}\right)\right)^2\\&=(\Delta r)^2\left(p_u^{(r)}+p_d^{(r)}-\left(p_u^{(r)}-p_d^{(r)}\right)^2\right).\end{aligned}$$

Furthermore,

$$E\left[R\frac{\Delta S}{S}\right]=(u-1)\left((\tilde{r}+\Delta r)p_{uu}+\tilde{r}p_{mu}+(\tilde{r}-\Delta r)p_{du}\right)+(d-1)\left((\tilde{r}+\Delta r)p_{ud}+\tilde{r}p_{md}+(\tilde{r}-\Delta r)p_{dd}\right)$$
$$=\tilde{r}\left((u-1)p_u^{(S)}+(d-1)p_d^{(S)}\right)+\Delta r\left((u-1)(p_{uu}-p_{du})+(d-1)(p_{ud}-p_{dd})\right),$$

and thus

$$Cov\left[R,\frac{\Delta S}{S}\right]=E\left[R\frac{\Delta S}{S}\right]-E[R]E\left[\frac{\Delta S}{S}\right]$$
$$=\Delta r\left\{(u-1)\left(p_{uu}-p_{du}-(p_u^{(r)}-p_d^{(r)})p_u^{(S)}\right)+(d-1)\left(p_{ud}-p_{dd}-(p_u^{(r)}-p_d^{(r)})p_d^{(S)}\right)\right\}.$$

Summarizing all this, we have in the case of Figure 5(a)

$$\rho\left[R,\frac{\Delta S}{S}\right]=\frac{(u-1)\left(p_{uu}-p_{du}-p_u^{(S)}(p_u^{(r)}-p_d^{(r)})\right)+(d-1)\left(p_{ud}-p_{dd}-p_d^{(S)}(p_u^{(r)}-p_d^{(r)})\right)}{\sqrt{\left(p_u^{(r)}+p_d^{(r)}-\left(p_u^{(r)}-p_d^{(r)}\right)^2\right)\left(u^2p_u^{(S)}+d^2p_d^{(S)}-\left(up_u^{(S)}+dp_d^{(S)}\right)^2\right)}}.$$

Similarly, in the case of Figure 5(b), we get

$$E[R]=(\tilde{r}+2\Delta r)p_u^{(r)}+(\tilde{r}+\Delta r)p_m^{(r)}+\tilde{r}p_d^{(r)}$$
$$=\tilde{r}+\Delta r\left(2p_u^{(r)}+p_m^{(r)}\right),$$
$$Var[R]=(\tilde{r}+2\Delta r)^2p_u^{(r)}+(\tilde{r}+\Delta r)^2p_m^{(r)}+\tilde{r}^2p_d^{(r)}-\left(\tilde{r}+\Delta r\left(2p_u^{(r)}+p_m^{(r)}\right)\right)^2$$
$$=(\Delta r)^2\left(4p_u^{(r)}+p_d^{(r)}-\left(2p_u^{(r)}+p_m^{(r)}\right)^2\right),$$
$$E\left[R\frac{\Delta S}{S}\right]=(u-1)\left((\tilde{r}+\Delta r)p_{uu}+(\tilde{r}+2\Delta r)p_{mu}+\tilde{r}p_{du}\right)+(d-1)\left((\tilde{r}+\Delta r)p_{ud}+(\tilde{r}+2\Delta r)p_{md}+\tilde{r}p_{dd}\right)$$
$$=\tilde{r}\left((u-1)p_u^{(S)}+(d-1)p_d^{(S)}\right)+\Delta r\left((u-1)(2p_{uu}+p_{mu})+(d-1)(2p_{ud}+p_{md})\right),$$

and

$$Cov\left[R,\frac{\Delta S}{S}\right]=\Delta r\left\{(u-1)\left(-p_u^{(S)}(2p_u^{(r)}+p_m^{(r)})+2p_{uu}+p_{mu}\right)+(d-1)\left(-p_d^{(S)}(2p_u^{(r)}+p_m^{(r)})+2p_{ud}+p_{md}\right)\right\}.$$

Thus, in the case of Figure 5(b), we get

$$\rho\left[R,\frac{\Delta S}{S}\right]=\frac{(u-1)\left(-p_u^{(S)}(2p_u^{(r)}+p_m^{(r)})+2p_{uu}+p_{mu}\right)+(d-1)\left(-p_d^{(S)}(2p_u^{(r)}+p_m^{(r)})+2p_{ud}+p_{md}\right)}{\sqrt{\left(4p_u^{(r)}+p_m^{(r)}-\left(2p_u^{(r)}+p_m^{(r)}\right)^2\right)\left(u^2p_u^{(S)}+d^2p_d^{(S)}-\left(up_u^{(S)}+dp_d^{(S)}\right)^2\right)}}.$$

Also, for the case of Figure 5(c),

$$E[R]=\tilde{r}p_u^{(r)}+(\tilde{r}-\Delta r)p_m^{(r)}+(\tilde{r}-2\Delta r)p_d^{(r)}$$
$$=\tilde{r}-\Delta r\left(p_m^{(r)}+2p_d^{(r)}\right),$$
$$Var[R]=\tilde{r}^2p_u^{(r)}+(\tilde{r}-\Delta r)^2p_m^{(r)}+(\tilde{r}-2\Delta r)^2p_d^{(r)}-\left(\tilde{r}-\Delta r\left(p_u^{(r)}+2p_d^{(r)}\right)\right)^2$$
$$=(\Delta r)^2\left(p_m^{(r)}+4p_d^{(r)}-\left(p_m^{(r)}+2p_d^{(r)}\right)^2\right),$$
$$E\left[R\frac{\Delta S}{S}\right]=(u-1)\left(\tilde{r}p_{uu}+(\tilde{r}-\Delta r)p_{mu}+(\tilde{r}-2\Delta r)p_{du}\right)+(d-1)\left(\tilde{r}p_{ud}+(\tilde{r}-\Delta r)p_{md}+(\tilde{r}-2\Delta r)p_{dd}\right)$$
$$=\tilde{r}\left((u-1)p_u^{(S)}+(d-1)p_d^{(S)}\right)-\Delta r\left((u-1)(p_{mu}+2p_{du})+(d-1)(p_{md}+2p_{dd})\right),$$

and

$$Cov\left[R,\frac{\Delta S}{S}\right]=\Delta r\left\{(u-1)\left(p_u^{(S)}(p_m^{(r)}+2p_d^{(r)})-p_{mu}-2p_{du}\right)+(d-1)\left(p_d^{(S)}(p_m^{(r)}+2p_d^{(r)})-p_{md}-2p_{dd}\right)\right\}.$$

Finally, in the case of Figure 5(c), we have

$$\rho\left[R,\frac{\Delta S}{S}\right]=\frac{(u-1)\left(p_u^{(S)}(p_m^{(r)}+2p_d^{(r)})-p_{mu}-2p_{du}\right)+(d-1)\left(p_d^{(S)}(p_m^{(r)}+2p_d^{(r)})-p_{md}-2p_{dd}\right)}{\sqrt{\left(p_m^{(r)}+4p_d^{(r)}-\left(p_m^{(r)}+2p_d^{(r)}\right)^2\right)\left(u^2p_u^{(S)}+d^2p_d^{(S)}-\left(up_u^{(S)}+dp_d^{(S)}\right)^2\right)}}.$$

8.2 Formulas for expected values and variances

The formulas of the transition probabilities in the tribinomial tree use some abbreviations for the expected values and variances for the different types of branching. We state their full formulas here:

$$\mu_{(a)}(R^2):=E_{(a)}[R^2]=(\tilde{r}+\Delta r)^2p_u^{(r)}+\tilde{r}^2p_m^{(r)}+(\tilde{r}-\Delta r)^2p_d^{(r)}$$

$$\mu_{(b)}\left(R^2\right) := E_{(b)}\left[R^2\right] = \left(\tilde{r} + 2\Delta r\right)^2 p_u^{(r)} + \left(\tilde{r} + \Delta r\right)^2 p_m^{(r)} + \tilde{r}^{\,2} p_d^{(r)}$$

$$\mu_{(c)}\left(R^2\right) := E_{(c)}\left[R^2\right] = \tilde{r}^{\,2} p_u^{(r)} + \left(\tilde{r} - \Delta r\right)^2 p_m^{(r)} + \left(\tilde{r} - 2\Delta r\right)^2 p_d^{(r)}$$

$$\sigma^2_{(a)}(R) := Var_{(a)}[R] = (\Delta r)^2 \left(p_u^{(r)} + p_d^{(r)} - \left(p_u^{(r)} - p_d^{(r)}\right)^2\right)$$

$$\sigma^2_{(b)}(R) := Var_{(c)}[R] = (\Delta r)^2 \left(4 p_u^{(r)} + p_d^{(r)} - \left(2 p_u^{(r)} + p_d^{(r)}\right)^2\right)$$

$$\sigma^2_{(c)}(R) := Var_{(c)}[R] = (\Delta r)^2 \left(p_m^{(r)} + 4 p_d^{(r)} - \left(p_m^{(r)} + 2 p_d^{(r)}\right)^2\right)$$

$$\sigma^2\left(\frac{\Delta S}{S}\right) := Var\left[\frac{\Delta S}{S}\right] = u^2 p_u^{(S)} + d^2 p_d^{(S)} - \left(u p_u^{(S)} + d p_d^{(S)}\right)^2 .$$