The Impact of Policyholder Behavior on Pricing, Hedging, and Hedge Efficiency of Withdrawal Benefit Guarantees in Variable Annuities

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Abstract

We analyze the impact of policyholder behavior on pricing, hedging and hedge efficiency of variable annuities with guaranteed lifetime withdrawal benefits. We consider different product designs, market models and approaches for modeling policyholder behavior in our analyses, covering deterministic behavior, behavior depending on the 'moneyness' of the guarantee, and optimal (value maximizing) behavior. First, we assess the risk of mispricing the guarantee due to inaccurate assumptions regarding future policyholder behavior. Comparing products with different ratchet mechanisms, we find that this potential for mispricing is the smallest for the product design with the most valuable ratchet mechanism. We further quantify the impact of different behavior models on the efficiency of the insurer's hedging strategy and the risk that results if the insurer's assumption for policyholder behavior deviates from actual behavior. Our analyses indicate significant differences between the considered products in terms of hedgeability and the sensitivity of the guarantee's value towards policyholder behavior and towards changes in the underlying asset's volatility. Also, we show that a simple path-dependent behavior model may not be suitable to fully assess the risk arising from policyholder behavior.

Keywords

Variable Annuities, Guaranteed Lifetime Withdrawal Benefits, Policyholder Behavior, Pricing, Hedging, Hedge Performance, Model Risk

1 Introduction

Variable annuities are fund-linked annuities where the policyholder typically pays a single premium into the policy and the money is then invested in one or several mutual funds. Variable annuities usually offer a wide range of investment options for the policyholder to choose from. On top of this basic structure, certain guarantee riders are offered by the insurer, adding different types of financial protection to the contract. There are several types of guarantee riders that come with variable annuities, including guaranteed minimum death benefits (GMDB) as well as guaranteed minimum living benefits, which can be categorized into three main subcategories: guaranteed minimum accumulation benefits (GMAB), guaranteed minimum income benefits (GMIB) and guaranteed minimum withdrawal benefits (GMWB). A GMAB guarantee provides the policyholder with some guaranteed value at one or several future points in time, while the GMIB guarantee provides a guaranteed annuity benefit, starting after a certain deferment period. With the GMWB rider, if certain conditions are met, the policyholders may continue to withdraw money from their account, even after the value of the account has dropped to zero. Such withdrawals are guaranteed as long as both, the amount that is withdrawn within each policy year and the total amount that is withdrawn over the term of the policy, stay within certain limits.

Insurers also started to include additional features in GMWB products. The most prominent is called "GMWB for Life" (also known as guaranteed lifetime withdrawal benefits, GLWB). With this type of guarantee, the total amount of withdrawals is unlimited. However, the annual amount that may be withdrawn while the insured is still alive may not exceed some maximum value; otherwise the guarantee will be affected. The withdrawals made by the policyholder are deducted from their account value as long as this value is positive. Afterwards, the insurer has to provide the guaranteed withdrawals for the rest of the insured's life. In return for this guarantee, the insurer receives guarantee charges, which are deducted from the policyholder's account value (as long as this value is positive). These charges are typically calculated as a fixed annual percentage of the so-called withdrawal benefit base (explained below) or of the account value. In a few products, annual guarantee charges are calculated as a fixed percentage of the single premium. In contrast to a conventional annuity, where the assets covering the liabilities are owned by the pool of insured, in a GLWB policy, the fund units of the contract are owned by the individual policyholder and remain accessible to the policyholder even in the payout phase. The policyholder may access the remaining fund assets at any time by (partially) surrendering the contract. In case of death of the insured, any remaining fund value (or a guaranteed minimum death benefit if such a rider is included and the corresponding value exceeds the fund value) is paid out to the beneficiary.

From an insurer's point of view, such products contain an interesting and challenging combination of several risks, resulting from policyholder behavior (with regard to surrender and withdrawal), financial markets, and longevity, alongside a variety of additional risks that come with most insurance contracts (e.g. operational and reputational risk). This combination of risks makes these guarantees challenging to hedge and has been in the focus of both, academics and practitioners.

Policyholder behavior risk stems from the fact that variable annuities usually offer the policyholder many choices, e.g. surrender, partial surrender, the decision whether or not and when to annuitize (in GMIB products) or the decision whether or not and how much to withdraw each year (in GMWB products). Several authors (cf. e.g. Milevsky and Salisbury, 2006, or Bauer et al., 2008) come to the conclusion that insurers assume what they call "suboptimal" policyholder behavior when pricing the guarantees. This means that (at least some) policyholders are assumed to not behave in a way that would maximize the value of the insurer's liabilities arising from the financial guarantees embedded in the products. From an insurer's risk management perspective, "optimal" policyholder behavior in this sense would constitute a worst-case scenario with respect to policyholder behavior. Bauer et al. (2008) state in particular that the value of certain guarantees under optimal policyholder behavior significantly exceeds typical prices charged in many insurance markets, whereas the value of the same guarantees assuming suboptimal behavior (using e.g. typical surrender probabilities and independence between surrender behavior and financial markets) are in line with observed prices. This appears to bear significant risks for the insurers. There are several examples where insurance companies had to update their policyholder behavior assumptions leading to significant increases in liabilities, see e.g. ING (2011), Manulife Financial (2011), and Sun Life Financial (2011). Other insurers even completely stopped their variable annuity business in certain markets, cf. for instance The Hartford (2009).

The effect of policyholder behavior not only on pricing but also – and much more importantly – on hedging and hedge efficiency of variable annuity guarantees should therefore be of interest to academics, product providers and regulators. The impact of policyholder behavior on the pricing of guarantees embedded in insurance contracts has been analyzed by several authors, e.g. by Grosen and Jørgensen (2000), Steffensen (2002), Bacinello et al. (2003, 2005, 2011) and Gao and Ulm (2012) and with focus on the optimal stopping time within the context of GMWB guarantees for example by Chen et al. (2008) and Yang and Dai (2013). Bernard et al. (2014) analyze optimal policyholder behavior for variable annuities with a GMAB. De Giovanni (2010) uses a 'Rational Expectation' model describing the policyholder's behavior in surrendering the contract, which also allows for irrational policyholder behavior. Knoller et al. (2013) analyze individual policy data from a Japanese variable annuity product and find evidence that confirms their "moneyness hypothesis": In their statistical analysis the fund performance and hence the value of the financial options and guarantees has the largest explanatory power for the surrender rate. They find that surrender rates increase with decreasing value of the guarantee and that policyholders' apparent rationality increases with increasing contract volume.

To our knowledge, there exists no simultaneous analysis of the impact of policyholder behavior on the pricing, hedging and hedge efficiency of GLWB riders with particular emphasis on different product designs. The present paper fills this gap: We extend the model presented in Kling et al. (2011) to incorporate non-deterministic policyholder behavior and – for different product designs – analyze the impact policyholder behavior has on pricing, hedging and hedge efficiency, and how results change if the capital market model incorporates stochastic instead of deterministic equity volatility.

The remainder of this paper is organized as follows. In Section 2, we describe our model framework that consists of three parts: the financial model, where for the sake of comparison we use both, the classic Black-Scholes model (with deterministic equity volatility) and the Heston model for the evolution of an underlying with stochastic equity volatility; the liability model that describes the different considered variable-annuity contracts with different GLWB options; and the valuation framework including the policyholder-behavior model, which allows for different policyholder strategies with regard to surrendering the contract. We particularly consider "optimal" policyholder behavior, as well as several "suboptimal" strategies, where, in both cases, "optimal" as explained above denotes the behavior that maximizes the value of the insurer's liabilities. In Section 3, we present the results of our analyses regarding the pricing of the guarantee. In particular, we analyze the differences in the option value for different product designs and how the option value depends on assumed policyholder behavior. This is a first indication for an insurer's potential loss arising from an inaccurate assessment of policyholder behavior. Section 4 deals with hedging strategies and hedge efficiency. Here, we particularly analyze how the insurer's expected profit and risk change if actual policyholder behavior deviates from the behavior assumed within the hedging strategy. Finally, Section 5 concludes.

2 Model Framework

In Bauer et al. (2008), a general framework for modeling and valuation of variable annuity contracts was introduced. Within this framework, any contract with one or several living benefit guarantees and/or a guaranteed minimum death benefit can be represented. In their numerical analysis however, only contracts with a rather short finite time horizon were considered. Holz et al. (2012) describe how GLWB products can be included in this model. In what follows, we apply the general framework of Bauer et al. (2008). However, in our concrete specification, additionally to the simple Black-Scholes model used in Bauer et al. (2008), we also consider a model which allows for stochastic equity volatility (Section 2.1). In Section 2.2, we introduce and define the specific product designs considered within our numerical analyses. Different models for policyholder behavior are introduced in Section 2.3, where also our valuation approach is summarized.

2.1 Financial Market

The valuation framework in this section follows in some parts the one used in Bacinello et al. (2010) and in others Bauer et al. (2008). We take as given a filtered probability space (Ω, Σ, F, P) in which P is the real-world (or physical) probability measure and $F \doteq (F_t)_{t\geq 0}$ is a filtration with $F_0 = \{\varnothing, \Omega\}$ and $F_t \subset \Sigma\, \forall\, t \geq 0$. We assume that trading takes place continuously over time and without any transaction costs or spreads. Furthermore, we assume that the price processes of the traded assets in the market are adapted and of bounded variation. For our analyses we assume two primary tradable assets: the underlying fund (or basket of funds), whose spot price at time t will be denoted by S_t, and the money-market account, whose value at time t will be denoted by B_t. We assume the money-market account to evolve at a constant risk-free rate of interest r:

$$\begin{aligned} & dB_t = rB_t dt \\ & \Rightarrow B_t = B_0 \exp(rt) \end{aligned} \tag{1}$$

For the dynamics of S_t, we use two different models. First, we assume the equity volatility to be deterministic and constant over time, and hence use the Black-Scholes model for our simulations. To allow for a more realistic equity volatility model, we also use the Heston model, in which both, the underlying and its (instantaneous) variance, are stochastic processes. These two models will be explained in the following two subsections.

2.1.1 Black-Scholes Model

In the Black-Scholes (1973) model, the underlying's spot price S_t follows a geometric Brownian motion whose dynamics under the real-world measure P are given by

$$dS_t = \mu S_t dt + \sigma_{BS} S_t dW_t, \quad S_0 \geq 0 \; , \tag{2}$$

where μ is the (constant) drift of the underlying, σ_{BS} its constant volatility and W_t denotes a P-Brownian motion. By Itō's lemma, S_t has the solution

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma_{BS}^2}{2}\right)t + \sigma_{BS} W_t\right), \quad S_0 \geq 0 \; . \tag{3}$$

2.1.2 Heston Model

There are various extensions to the Black-Scholes model that allow for a more realistic modeling of the underlying's volatility. We use the Heston (1993) model in our analyses where the instantaneous (or local) volatility of the asset is stochastic. Under the Heston model, the market is assumed to be driven by two stochastic processes: the underlying's price S_t, and its instantaneous variance V_t, which is assumed to follow a one-factor square-root process identical to the one used in the Cox-Ingersoll-Ross interest rate model (Cox et al., 1985). The dynamics of the two processes under the real-world measure P are given by the following system of stochastic differential equations:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^1, \quad S_0 \geq 0 \tag{4}$$
$$dV_t = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t} dW_t^2, \quad V_0 \geq 0, \tag{5}$$

where μ again is the drift of the underlying, V_t is the local variance at time t, κ is the speed of mean reversion, θ is the long-term variance, σ_v is the so-called "volatility of volatility", and $W_t^{1/2}$ are correlated P-Brownian motion processes (with correlation parameter ρ). The condition $2\kappa\theta \geq \sigma_v^2$ ensures that the variance process will remain strictly positive almost surely (see Cox et al., 1985).

2.1.3 Equivalent Martingale Measure

Assuming the absence of arbitrage opportunities in the financial market, there exists a probability measure Q that is equivalent to P and under which the gain from holding a traded asset is a Q-martingale after discounting with the chosen numéraire process, in our case the money-market account. Q is called equivalent martingale measure. While – under the usual assumptions – the transformation to such a measure is unique under the Black-Scholes model (cf. e.g. Bingham and Kiesel, 2004), it is not under the Heston model. Within the Heston model, since there are two sources of risk, there are also two market-price-of-risk processes, denoted by γ_t^1 and γ_t^2 (corresponding to W_t^1 and W_t^2). Heston (1993) proposed the following restriction on the market price of volatility risk process, assuming it to be linear in volatility,

$$\gamma_t^1 = \lambda \sqrt{V_t} \quad . \tag{6}$$

Provided both measures, P and Q, exist, the Q-dynamics of S_t and V_t, again under the assumption that no dividends are paid, are then given by

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^{Q,1}, \quad S_0 \geq 0 \tag{7}$$

$$dV_t = \kappa^* \left(\theta^* - V_t\right) dt + \sigma_v \sqrt{V_t} dW_t^{Q,2}, \quad V_0 \geq 0 \tag{8}$$

where $W_t^{Q,1}$ and $W_t^{Q,2}$ are two correlated Q-Brownian motion processes (with correlation parameter ρ) and where

$$\kappa^* = \left(\kappa + \lambda\sigma_v\right), \quad \theta^* = \frac{\kappa\theta}{\left(\kappa + \lambda\sigma_v\right)} \tag{9}$$

are the risk-neutral counterparts to κ and θ (cf., for instance, Wong and Heyde, 2006).

2.2 Model of the Liabilities

With variable annuities, the single premium P is invested in one or several mutual funds. We call the value of the policyholder's individual portfolio the *account value* and denote its value at time t by AV_t. All charges are taken from the account value by cancellation of fund units. Furthermore, the policyholder has the possibility to surrender the contract or to withdraw a portion of the account value.

Products with a GMWB option give the policyholder the possibility to perform guaranteed withdrawals. In this paper, we focus on the case where such withdrawals are guaranteed lifelong ("GMWB for Life" or guaranteed lifetime withdrawal benefits, GLWB). The initially guaranteed withdrawal amount is usually a certain percentage x_{WL} of the single premium P. In most products, x_{WL} depends on the age when withdrawals start. Any remaining account value at the time of death is paid to the beneficiary as death benefit. If, however, the account value of the policy drops to zero while the insured is still alive, the policyholder can still continue to withdraw the guaranteed amount until death of the insured. The insurer charges a fee for this guarantee, which is usually a pre-specified annual percentage of the withdrawal benefit base, the account value or the single premium. In what follows, we will assume that the guarantee charge is a percentage of the account value and that withdrawals may only occur on the policy's anniversary dates.

Often, GLWB products contain certain features that lead to an increase of the guaranteed withdrawal amount if the underlying funds perform well. Typically, on every policy anniversary, the current account value is compared to a certain reference value, which we refer to as 'withdrawal benefit base'. Whenever the account value exceeds the withdrawal benefit base, the guaranteed annual withdrawal amount is increased (step-up or ratchet). In our numerical analyses in Sections 3 and 4, we consider three different product designs that can be observed in the market:

- **No Ratchet (Product I)**: The first and simplest alternative is one where no ratchets exist at all. In this case, the guaranteed annual withdrawal amount is constant and does not depend on market movements.

- **Lookback Ratchet (Product II)**: The second alternative is a ratchet mechanism where the withdrawal benefit base at outset is given by the single premium paid. During the contract term, on each policy anniversary date, the withdrawal benefit base is increased to the account value, if the account value exceeds the previous withdrawal benefit base. The guaranteed annual withdrawal amount is increased accordingly to x_{WL} multiplied by the new withdrawal benefit base. This effectively means that the fund performance needs to compensate for charges and annual withdrawals in order to increase future guaranteed withdrawals. Increases in the guaranteed withdrawal amount are permanent, i.e. over time, the guaranteed withdrawal amount may only increase, never decrease.

- **Remaining WBB Ratchet (Product III)**: The basic idea of the third product is to provide a ratchet mechanism where, in order to increase guaranteed annual withdrawals, the fund performance needs to compensate only for charges, but not for annual withdrawals. In this product, the withdrawal benefit base at outset is also given by the single premium paid. However, at each withdrawal date, the withdrawal benefit base is reduced by the withdrawn amount (if this amount does not exceed the guaranteed withdrawal amount). If on a policy anniversary the current account value exceeds this reduced withdrawal benefit base by a certain amount Δ, the guaranteed annual withdrawal is increased by $x_{WL} \cdot \Delta$. After such an increase, the withdrawal benefit base is reset to the account value. This ratchet mechanism is therefore c.p. somewhat “richer” than the Lookback Ratchet. As a consequence, the initially guaranteed withdrawal amount should c.p. be lower than with a product offering a Lookback Ratchet. As with the Lookback Ratchet design, increases in the guaranteed amount are permanent.

Throughout the paper, we assume that administration charges and guarantee charges are deducted at the end of each policy year as a percentage φ^{adm} and φ^{guar} of the account value. Additionally, we allow for upfront acquisition charges φ^{acq} as a percentage of the single premium P. This leads to $AV_0 = P \cdot \left(1-\varphi^{acq}\right)$.

We denote the guaranteed withdrawal amount at time t by W_t^{guar} and the corresponding withdrawal benefit base by WBB_t. At inception, for each of the considered products, the initial withdrawal benefit base is set to P and hence the guaranteed withdrawal amount for the initial withdrawal is given by $W_0^{guar+} = x_{WL} \cdot WBB_0 = x_{WL} \cdot P$. The amount actually withdrawn by the client is denoted by W_t.[1]

Since we restrict our analyses to single premium contracts, policyholder actions during the life of the contract are limited to withdrawals and (partial) surrender.

During the year, all processes are subject to capital market movements. As mentioned above, we allow for withdrawals at policy anniversaries only. Also, we assume that death benefits are paid out at policy anniversaries if the insured person has died during the previous year. Thus, at each policy anniversary $t = 1,2,\ldots,T$, we have to distinguish between the value of a variable $(\cdot)_t^-$ immediately before and the value $(\cdot)_t^+$ after withdrawals, (partial) surrender, and death

[1] Note that the client can choose to withdraw less than the guaranteed amount, thereby increasing the probability of future ratchets. If the client wants to withdraw more than the guaranteed amount, any exceeding withdrawal would be considered a partial surrender.

benefit payments. For the latter, we assume that no additional guaranteed minimum death benefit rider is included in the policy, i.e. in case of death the remaining fund value is paid out.

In what follows, in the spirit of Bauer et al. (2008), we describe the development between two policy anniversaries and the transition at policy anniversaries for the considered contract designs. From these, we are finally able to determine all benefits for any given policyholder strategy and any capital market path. This allows for an analysis of such contracts in a Monte-Carlo framework.

2.2.1 Development between two Policy Anniversaries

We assume that the annual fees φ^{adm} and φ^{guar} are deducted from the policyholder's account value at the end of each policy year. Thus, the development of the account value between two policy anniversaries is given by

$$AV_{t+1}^{-} = AV_t^{+} \cdot \frac{S_{t+1}}{S_t} \cdot \exp\left(-\varphi^{adm} - \varphi^{guar}\right). \tag{10}$$

At the end of each year, the different ratchet mechanisms are applied after deduction of charges and before any other actions are taken. Thus W_t^{guar} develops as follows:

- **No Ratchet:** $WBB_{t+1}^{-} = WBB_t^{+} = P$ and $W_{t+1}^{guar-} = W_t^{guar+} = x_{WL} \cdot P$.
- **Lookback Ratchet:** $WBB_{t+1}^{-} = \max\left\{WBB_t^{+}, AV_{t+1}^{-}\right\}$ and $W_{t+1}^{guar-} = x_{WL} \cdot WBB_{t+1}^{-} = \max\left\{W_t^{guar+}, x_{WL} \cdot AV_{t+1}^{-}\right\}$.
- **Remaining WBB Ratchet**: Since withdrawals are only possible on policy anniversaries, the withdrawal benefit base during the year develops like in the Lookback Ratchet case. Thus, we have $WBB_{t+1}^{-} = \max\left\{WBB_t^{+}, AV_{t+1}^{-}\right\}$ and $W_{t+1}^{guar-} = W_t^{guar+} + x_{WL} \cdot \max\left\{AV_{t+1}^{-} - WBB_t^{+}, 0\right\}$.

2.2.2 Transition at a Policy Anniversary *t*

At the policy anniversaries, we have to distinguish the following four cases:

a) The insured has died within the previous year (t-1,t]

If the insured has died within the previous policy year, the account value is paid out as death benefit. With the payment of the death benefit, the insurance contract matures. Thus, $AV_t^{+} = 0$, $WBB_t^{+} = 0$, $W_t = 0$, and $W_t^{guar+} = 0$.

b) The insured has survived the previous policy year and does not withdraw any money from the account at time *t*

If no death benefit is paid out to the policyholder and no withdrawals are made from the contract, i.e. $W_t = 0$, we get $AV_t^{+} = AV_t^{-}$, $WBB_t^{+} = WBB_t^{-}$, and $W_t^{guar+} = W_t^{guar-}$.

c) The insured has survived the previous policy year and at the policy anniversary withdraws an amount within the limits of the withdrawal guarantee

If the insured has survived the past year, no death benefits are paid. Any withdrawal W_t up to the guaranteed annual withdrawal amount W_t^{guar-} reduces the account value by the withdrawn amount. Of course, we do not allow for negative policyholder account values and thus get $AV_t^+ = \max\{0; AV_t^- - W_t\}$.

For the alternatives "No Ratchet" and "Lookback Ratchet", the withdrawal benefit base and the guaranteed annual withdrawal amount remain unchanged, i.e. $WBB_t^+ = WBB_t^-$, and $W_t^{guar+} = W_t^{guar-}$. For the alternative "Remaining WBB Ratchet", the withdrawal benefit base is reduced by the withdrawal taken, i.e. $WBB_t^+ = \max\{0; WBB_t^- - W_t\}$ and the guaranteed annual withdrawal amount remains unchanged, i.e. $W_t^{guar+} = W_t^{guar-}$.

d) The insured has survived the previous policy year and at the policy anniversary withdraws an amount exceeding the limits of the withdrawal guarantee

In this case again, no death benefits are paid. For the sake of brevity, we only give the formulas for the case of full surrender, since partial surrender is not analyzed in what follows.[2] In case of full surrender, the complete account value is withdrawn. We then set $AV_t^+ = 0$, $WBB_t^+ = 0$, $W_t = AV_t^-$, and $W_t^{guar+} = 0$ and the contract terminates. However, the policyholder does not receive the full asset value as surrender benefit, since surrender fees φ_t^{surr} are deducted from the cash amount exceeding the guaranteed withdrawal amount.

2.3 Valuation

Let Q be an equivalent martingale measure of the financial market (cf. section 2.1.3). Assuming independence between financial markets and mortality as well as risk-neutrality of the insurer with respect to mortality and behavioral risk, we are able to use the product measure of Q and the mortality measure. In what follows, we denote this measure by $\hat{Q}$.

As mentioned earlier, for the contracts considered within our analyses, policyholder actions are limited to withdrawals and (partial) surrender. In our numerical analyses in Sections 3 and 4, we only consider two possible policyholder actions: withdrawal of the guaranteed withdrawal amount, i.e. $W_t = W_t^{guar-}$, or full surrender, i.e. $W_t = AV_t^-$. This also means that we assume that withdrawals begin at the earliest anniversary possible and, hence, that there is no initial waiting period before the first withdrawal. To keep notation simple, we only give formulas for the considered cases (cf. Bauer et al. (2008) for formulas for the other cases).

We denote by x_0 the insured's age at the start of the contract, ${}_t p_{x_0}$ the probability under $\hat{Q}$ for a x_0-year old to survive the next t years, q_{x_0+t} the probability under $\hat{Q}$ for a (x_0+t)-year

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[2] For details on partial surrender, we refer the reader to Bauer et al. (2008).

old to die within the next year, and let ω be the limiting age of the mortality table, i.e. the age beyond which survival is deemed impossible. The probability under $\hat{Q}$ that an insured aged x_0 at inception passes away in the year $(t,t+1]$ is thus given by ${}_t p_{x_0} \cdot q_{x_0+t}$. The limiting age ω allows for a finite time horizon $T = \omega - x_0 + 1$.

For pricing purposes, we consider a pool of policyholders who hold identical contracts and in which each insured has the same age, same gender and same mortality probability. We assume the number of policyholders to be large enough such that the assumption that deaths occur exactly according to the probabilities q_{x_0+t} is justified. The policyholders in the pool may however differ in their (surrender) behavior.

We model the (surrender) behavior of the policyholders in the pool as a $\hat{F}$-adapted family of random variables $\xi = (\xi_t)_{t=1,...,T}$, where $0 \le \xi_t \le 1, \forall\, t = 1,...,T$, represents the fraction of the remaining policyholders at time t who surrender their contract at time t. After the guarantee has been triggered, i.e. $W_t^{guar-} > AV_t^-$ for some $t=1,...,T$, there is no rational reason for a policyholder to surrender their contract, hence we set $\xi_t = 0, \forall\, t \ge \tau_G$, where τ_G represents a $\hat{F}$-stopping time indicating the policy anniversary at which the guarantee of the GLWB rider triggers, i.e. the smallest $t=1,...,T$ for which $W_t^{guar-} > AV_t^-$ holds. If the guarantee does not trigger during the contract's lifetime, we set $\tau_G = T$.

For a given behavior assumption $\xi = (\xi_t)_{t=1,...,T}$, all contractual cash flows of the pool of policies are specified for any given capital market scenario. Thus both, the guarantee payments (i.e. payments made by the insurer after the account value has dropped to zero) at times $i \in \{1,2,...,T\}$, denoted by $G_i^P(\xi)$, and the guarantee fee payments $G_i^F(\xi)$ made by the policyholder (including surrender fee payments), again at times $i \in \{1,2,...,T\}$, are known. For any given ξ, the time-t value $V_t^G(\xi)$ of the GLWB rider is then given by the expected present value of all future guarantee payments $G_i^P(\xi)$, $i \in \{1,2,...,T\}$, minus future guarantee fees $G_i^F(\xi)$, $i \in \{1,2,...,T\}$,

$$V_t^G(\xi) = E_{\hat{Q}}\left[\sum_{i=t+1}^{T} e^{-r(i-t)} \left(G_i^P(\xi) - G_i^F(\xi) \right) \middle| \hat{F}_t \right]. \tag{11}$$

In the following numerical section, this value is calculated using (nested) Monte-Carlo simulations.

Within our numerical analyses, we consider five different assumptions regarding policyholder behavior:

a) No surrender

Under this assumption, policyholders never surrender their contract, i.e.

$$\xi^0 = \left(\xi_t^0\right)_{t=1,\ldots,T},$$
$$\xi_t^0 = 0, \quad \forall\, t = 1,\ldots,T.$$

b) Deterministic surrender

Surrender under this assumption occurs according to pre-specified deterministic (time-dependent) percentages $\left(s_t\right)_{t=1,\ldots,T}$, $0 \le s_t \le 1, \forall\, t = 1,\ldots,T$, as long as the guarantee has not been triggered. In formulas,

$$\xi^d = \left(\xi_t^d\right)_{t=1,\ldots,T},$$
$$\xi_t^d := \begin{cases} s_t, & 1 \le t < \tau_G \\ 0, & else \end{cases}$$

c) Longstaff-Schwartz approximation to optimal surrender

In order to compute the fair value of an American option using Monte-Carlo techniques, Longstaff and Schwartz (2001) introduced a method in which optimal behavior is approximated via least-squares regression of the conditional expectation of the option's payoff, given some path- and time-dependent variables. Essentially, when applied to pricing of the GLWB rider, their algorithm works as follows:

1. Define a set of base functions that take some state variables of the contract and the scenario as argument and return a real number.
2. Create a set of *N* scenarios under $\hat{Q}$.
3. Starting at *T*-1, at each policy anniversary *t*, compute the present value of the cash flow between *t* and *T* for each scenario in which the guarantee has not been triggered at time *t*. Fit the linear least-squares regression with these present values as dependent variables and the base functions with the corresponding state variables of the contract and the scenario as input variables.
4. Evaluate the resulting approximation of the GLWB rider's continuation value for each scenario and decide whether the policyholder should surrender or not. If they surrender, the cash flow following *t* is set to zero and the cash flow at *t* to minus the surrender fee paid.
5. Repeat steps 3-5 for *t*-1 until *t*=0 is reached.

As base functions we use weighted Hermite polynomials up to a degree of three for each state variable and cross products hereof, again up to a degree of three, as well as a constant. Before simulation and/or pricing, we first execute the Longstaff-Schwartz algorithm with a separate set of scenarios in order to avoid an upward bias.

The surrender behavior of the policyholder can be considered optimal – in the sense that it maximizes the option value V_t^G of the GLWB rider – if the policyholder decides to surrender the contract whenever the benefit from discontinuing the contract (i.e. the negative of the continuation value) exceeds the surrender fees.

With $\hat{V}_t^G$ denoting the approximated continuation value of the GLWB rider at time *t*, the policyholder behavior is modeled as follows:

$$\xi^{LS} = \left(\xi_t^{LS}\right)_{t=1,\dots,T},$$
$$\xi_t^{LS} := \begin{cases} 1, & \hat{V}_t^G < -\varphi_t^{surr} AV_t^-,\ 1 \le t < \tau_G \\ 0, & else \end{cases}$$

In what follows, we refer to surrender behavior according to this algorithm as being "optimal", although we have to keep in mind that it is only an approximation for the value maximizing strategy defined as $\xi^* = \arg\max_{\xi \in \Xi} V_0^G(\xi)$, where Ξ denotes the set of all admissible strategies.

d) Function of moneyness

Within this approach, we model the fraction of the policyholders who surrender their contract as a function of time and the "in-the-moneyness" of the guarantee (as, for example, described in American Academy of Actuaries, 2005).

We define the moneyness θ_t of the guarantee at time *t* as the ratio of the surrender value (account value less surrender fees) and the 'strike price' of the guarantee, for which we use the net present value of an immediate annuity paying the current guaranteed withdrawal amount annually until the insured's death. Because this annuity's net present value is a lower limit for the sum of asset value and option value of the GLWB rider, θ_t will be upward biased and not reside around 1 ("at-the-money") as desired. To correct for this, we use θ_0, the moneyness at inception of the contract, as benchmark and use the relative deviation of θ_t hereof as measure.

The basis for the surrender function is a set of given pre-specified deterministic percentages $\left(s_t\right)_{t=1,\dots,T}$, $0 \le s_t \le 1, \forall\, t = 1,\dots,T$ (as in the deterministic surrender scenario). However, we now model the fraction of the surrendering policyholders at time *t* as s_t multiplied by a factor that depends on the moneyness-variable θ_t. In detail, we model the behavior according to the following formulas:

$$\xi^{ITM} = \left(\xi_t^{ITM}\right)_{t=1,\dots,T},$$
$$\xi_t^{ITM} := \begin{cases} s_t \cdot \eta_1(\theta_t / \theta_0), & 1 \le t < \tau_G \\ 0, & else \end{cases},$$
$$\eta_1(x) := \begin{cases} 1/3, & x < 0.95 \\ 1, & 0.95 \le x < 1.05 \\ 3, & 1.05 \le x < 1.15 \\ 5, & x \ge 1.15 \end{cases}$$

e) Function of option value

Here, we use a similar approach as for the 'function of moneyness', except that we now use the sum of the rider's (approximated) continuation value and the surrender charge as decision variable. Within the Longstaff-Schwartz algorithm, it is optimal for the policyholder to discontinue the contract whenever this value becomes negative. Using again pre-specified probabilities $(s_t)_{t=1,...,T}$, $0 \le s_t \le 1, \forall\, t = 1,...,T$, this modeling approach is defined as follows:

$$\xi^{OV} = \left(\xi_t^{OV}\right)_{t=1,...,T}$$

$$\xi_t^{OV} := \begin{cases} s_t \cdot \eta_2(\hat{V}_t^G(\xi^0) + \varphi_t^{surr} AV_t^-), & 1 \le t < \tau_G \\ 0, & else \end{cases}$$

$$\eta_2(x) := \begin{cases} 1/3, & x > 0.01 \\ 1, & 0.01 \ge x > -0.01 \\ 3, & -0.01 \ge x > -0.03 \\ 5, & x \le -0.03 \end{cases}$$

Note that the last two models for policyholder behavior, d) and e), allow for the following interpretation which appears to be the motivation for the use of such models in practice: If a certain percentage of the policyholders follow a more or less optimal strategy (in the sense that they intuitively or with the help of professional advisors aim at maximizing the value V_t^G of the embedded guarantee) and the rest of the policyholders are assumed to follow a suboptimal strategy with deterministic surrender rates, then a pool would show patterns similar to models d) and e).

3 Contract Analysis

3.1 Assumptions

For all of the analyses we use the fee structure given in Table 1.

Acquisition charges	4.00 % of single premium
Management charges	1.50 % p.a. of AV
Guarantee charges	1.50 % p.a. of AV

Table 1: Fee structure for the considered contracts.

We further assume the policyholder to be a 65 year old male. For pricing purposes, we use best-estimate annuitant mortality probabilities given in the DAV 2004R table published by the German Actuarial Society (DAV).

As described in Section 2.3, we use different assumptions for the policyholder behavior. In the case where surrender is assumed to be deterministic, we use the surrender pattern given in Table 2. As observed for many products in many markets, we assume higher surrender rates in earlier years and some base surrender in later years.

Year	Surrender rate p_t^S
1	6 %
2	5 %
3	4 %
4	3 %
5	2 %
≥ 6	1 %

Table 2: Assumed deterministic surrender rates.

Besides deterministic surrender (in what follows denoted by *DS*), we also analyze the other types of policyholder behavior introduced in Section 2.3, i.e. Longstaff-Schwartz-optimal surrender behavior (*optimal*), surrender behavior depending on the option value (*OV*), and surrender behavior depending on the "in-the-moneyness" of the option (*ITM*). We also consider the case without any surrender (*NS*).

3.2 Determination of the Fair Guaranteed Withdrawal Rate

For the pricing of the contract, i.e. for the determination of the guaranteed withdrawal rate x_{WL} that makes the contract fair at inception in the sense that $V_0^G = 0$ holds, we perform a root search with x_{WL} as argument and the value of the option V_0^G as function value, cf. e.g. Bauer et al. (2008) or Kling et al. (2011). In this process, V_0^G is computed via Monte-Carlo simulation, where 100,000 paths are used per valuation.

3.2.1 Results for the Black-Scholes model

In Table 3, we show the fair guaranteed withdrawal rates x_{WL} for different ratchet mechanisms, volatilities, rates of interest, surrender fees and policyholder-behavior assumptions. Note that we here analyze the impact of the policyholder behavior assumptions used for pricing the contract. Effects resulting from a potential deviation between actual policyholder behavior and behavior assumed in pricing and hedging will be analyzed in Sections 3.3 and 4.

σ_{BS}, r	Behavior	Ratchet mechanism: Product I (No Ratchet) φ_t^{surr} = 1%	Product I (No Ratchet) φ_t^{surr} = 3%	Product II (Lookback) φ_t^{surr} = 1%	Product II (Lookback) φ_t^{surr} = 3%	Product III (Remaining WBB) φ_t^{surr} = 1%	Product III (Remaining WBB) φ_t^{surr} = 3%
σ_{BS} = 15%, r = 4%	*Optimal*	4.75	4.95	4.69	4.77	4.45	4.45
	OV	5.13	5.23	4.86	4.90	4.51	4.53
	ITM	5.14	5.24	4.87	4.91	4.52	4.54
	NS	5.27	5.27	4.82	4.82	4.45	4.45
	DS	5.48	5.54	5.04	5.09	4.65	4.70
σ_{BS} = 20%, r = 4%	*Optimal*	4.23	4.49	4.17	4.26	4.02	4.02
	OV	4.75	4.87	4.36	4.40	4.09	4.10
	ITM	4.75	4.87	4.37	4.42	4.09	4.12
	NS	5.00	5.00	4.34	4.34	4.03	4.03
	DS	5.20	5.25	4.54	4.59	4.22	4.26
σ_{BS} = 22%, r = 4%	*Optimal*	4.03	4.31	3.96	4.06	3.84	3.86
	OV	4.59	4.72	4.17	4.21	3.92	3.93
	ITM	4.60	4.72	4.17	4.22	3.92	3.95
	NS	4.89	4.89	4.14	4.14	3.86	3.86
	DS	5.08	5.13	4.34	4.39	4.05	4.09
σ_{BS} = 25%, r = 4%	*Optimal*	3.74	4.04	3.67	3.78	3.59	3.61
	OV	4.37	4.51	3.88	3.92	3.67	3.69
	ITM	4.37	4.50	3.88	3.94	3.68	3.70
	NS	4.72	4.72	3.87	3.87	3.62	3.62
	DS	4.90	4.95	4.05	4.10	3.80	3.84
σ_{BS} = 22%, r = 2%	*Optimal*	3.11	3.33	3.08	3.18	3.06	3.10
	OV	3.56	3.67	3.28	3.32	3.16	3.18
	ITM	3.57	3.67	3.28	3.33	3.16	3.19
	NS	3.78	3.78	3.27	3.27	3.12	3.12
	DS	3.96	4.00	3.44	3.48	3.28	3.32
σ_{BS} = 22%, r = 3%	*Optimal*	3.55	3.80	3.51	3.61	3.45	3.48
	OV	4.06	4.17	3.71	3.75	3.53	3.55
	ITM	4.07	4.18	3.71	3.76	3.54	3.56
	NS	4.32	4.32	3.70	3.70	3.48	3.48
	DS	4.50	4.55	3.88	3.93	3.66	3.70
σ_{BS} = 22%, r = 5%	*Optimal*	4.53	4.85	4.44	4.55	4.24	4.25
	OV	5.16	5.30	4.65	4.69	4.31	4.32
	ITM	5.17	5.31	4.65	4.70	4.32	4.34
	NS	5.49	5.49	4.62	4.62	4.25	4.25
	DS	5.69	5.75	4.83	4.88	4.45	4.49

Table 3: Fair guaranteed withdrawal rates x_{WL} in percent under the Black-Scholes model for different ratchet mechanisms, policyholder behavior assumptions, volatilities, rates of interest and surrender fees.

Obviously, the product design without any ratchet allows for the highest withdrawal rates throughout, while the remaining WBB ratchet (which constitutes the 'richest' type of ratchet) allows for the lowest. It is also obvious that fair withdrawal rates are decreasing with increasing volatility and/or decreasing interest rates, since the corresponding guarantees increase in value with increasing volatility or decreasing interest rates. Our main focus, however, is on the analysis of different assumptions about policyholder behavior:

As defined in Section 2.3, optimal surrender behavior maximizes the value V_t^G of the contract. Thus, the fair withdrawal rates are the lowest in this case. For all considered parameter combinations, the assumption of deterministic surrender rates leads to the highest fair withdrawal rate. The difference between the fair withdrawal rates in these two cases can exceed a full percentage point. For a volatility of 25%, for example, and in the product without ratchet, the fair withdrawal rate assuming deterministic surrender amounts to 4.90% in the case of a surrender fee of 1% (and 4.95% for a surrender fee of 3%) while for optimal policyholder behavior, the fair withdrawal rate is only 3.74% (4.04%). Hence, an insurer assuming deterministic behavior would be willing to provide policyholders lifelong guaranteed withdrawal amounts that are (c.p.) more than 30% higher than the rates offered by a more conservative insurer assuming optimal policyholder behavior.

For the product design with no ratchet, this difference in withdrawal rates is increasing with increasing volatility and with increasing interest rates. Thus, the potential for mispricing resulting from too aggressive assumptions for policyholder behavior is also increasing. For the product designs with ratchet, the difference of the fair withdrawal rate assuming deterministic policyholder behavior and optimal policyholder behavior, respectively, is significantly smaller and much less sensitive to changes in volatility or interest rates. For the lookback ratchet, the difference is about 30 to 40 basis points, in the case of the remaining WBB ratchet, the difference amounts to 20 to 25 basis points. Thus, the potential for mispricing by assuming incorrect policyholder behavior is the smallest for the product design with the most valuable ratchet mechanism.

One reason for this can be seen by comparing the fair withdrawal rate assuming no surrender and optimal surrender. If the ratchet mechanism is quite valuable (i.e. remaining WBB ratchet), there is very little or even no difference in the corresponding fair withdrawal rates. Thus, no surrender seems to be very close to an optimal policyholder behavior. Hence, by assuming some deterministic (and fairly low) surrender rate, the assumption basically is that almost all policyholders behave optimally (by not surrendering) and only very few behave suboptimally. For a product design without any ratchet on the other hand, surrender can become optimal if funds perform well. In all these scenarios, deterministic surrender rates imply the assumption of a high portion of customers behaving suboptimally by not surrendering and only a low portion of customers displaying optimal behavior.

The two path-dependent assumptions about policyholder behavior (*OV* and *ITM*) show a rather similar pattern. For the product design without ratchet, both show a significant potential for mispricing. Even if volatility is only 15%, the difference in the fair withdrawal rates between path-dependent assumptions and optimal behavior is around 40 basis points (roughly 30 basis points for a surrender fee of 3%). Again, with increasing volatility or interest rates, this difference also increases. However, for this product design, the considered path-dependent assumptions lead to lower guaranteed withdrawal rates than assuming no surrender. Thus, in this case the potential for mispricing is lower if path-dependent assumptions are made. This changes if ratchets are included into the product. Then again, the differences in withdrawal rates decrease. At the same time, the fair withdrawal rates assuming no surrender are lower than the fair withdrawal rates assuming path-dependent surrender. Thus, within this modeling framework, even though an insurer assumes surrender behavior that is somehow linked to the option value or the in-the-moneyness of the option, the potential for mispricing is higher than when assuming no surrender.

Fair withdrawal rates obviously increase with increasing surrender fees. It is, however, worth noting that for product I the difference in fair withdrawal rates between a surrender fee of 3% and 1% increases with increasing "optimality" of the policyholders' behavior. If a strong ratchet mechanism is included (e.g. product III), however, there is almost no difference if policyholders behavior optimally. This again is due to the fact that for this product design, not surrendering is close to optimal, even for the lower surrender fee.

3.2.2 Results for the Heston model

For the Heston model, we use the model parameters given in Table 4, that were derived by Eraker (2004), and stated in annualized form for instance by Ewald et al. (2007).

Parameter	Numerical value
r	0.04
θ	$(0.22)^2$
κ	4.75
σ_V	0.55
ρ	-0.569
$V(0)$	θ

Table 4: Parameters for the Heston model.

One of the key parameters in the Heston model is the market price of volatility risk λ. Since absolute λ-values are hard to interpret, in the following table we show the values of the long-term variance and the speed of mean reversion for different values of λ.

Market price of volatility risk	Speed of mean reversion κ^*	Long-term variance θ^*
$\lambda = 3$	6.40	$(0.190)^2$
$\lambda = 2$	5.85	$(0.198)^2$
$\lambda = 1$	5.30	$(0.208)^2$
$\lambda = 0$	4.75	$(0.220)^2$
$\lambda = -1$	4.20	$(0.234)^2$
$\lambda = -2$	3.65	$(0.251)^2$
$\lambda = -3$	3.10	$(0.272)^2$

Table 5: *Q*-parameters for different values of the market price of volatility risk.

Higher values of λ correspond to a lower volatility and a higher mean-reversion speed, while lower (and negative) values of λ correspond to high volatilities and a lower speed of mean reversion. E.g., $\lambda = 2$ implies a long-term volatility of 19.8% and $\lambda = -2$ implies a long-term volatility of 25.1%.

In the following table, we show the fair annual guaranteed withdrawal rates under the Heston model for all different product designs using the same assumptions regarding policyholder behavior and interest rates as for the Black-Scholes model, and values of λ between -2 and 2.

		Ratchet mechanism					
		Product I (No Ratchet)		Product II (Lookback)		Product III (Remaining WBB)	
λ, r	Behavior	φ_t^{surr} = 1%	φ_t^{surr} = 3%	φ_t^{surr} = 1%	φ_t^{surr} = 3%	φ_t^{surr} = 1%	φ_t^{surr} = 3%
λ = 2, r = 4%	*Optimal*	4.26	4.54	4.20	4.32	4.04	4.04
	OV	4.77	4.90	4.41	4.45	4.11	4.12
	ITM	4.78	4.91	4.42	4.47	4.11	4.14
	NS	5.02	5.02	4.39	4.39	4.05	4.05
	DS	5.21	5.27	4.59	4.64	4.24	4.28
λ = 1, r = 4%	*Optimal*	4.17	4.46	4.10	4.22	3.96	3.96
	OV	4.70	4.83	4.32	4.36	4.02	4.04
	ITM	4.71	4.83	4.32	4.37	4.03	4.06
	NS	4.96	4.96	4.30	4.30	3.97	3.97
	DS	5.16	5.21	4.50	4.55	4.15	4.20
λ = 0, r = 4%	*Optimal*	4.06	4.36	3.99	4.11	3.86	3.87
	OV	4.61	4.75	4.21	4.25	3.93	3.95
	ITM	4.62	4.75	4.21	4.27	3.94	3.97
	NS	4.90	4.90	4.19	4.19	3.87	3.87
	DS	5.09	5.14	4.39	4.44	4.06	4.10
λ = -1, r = 4%	*Optimal*	3.95	4.24	3.86	3.98	3.75	3.76
	OV	4.52	4.66	4.08	4.13	3.82	3.84
	ITM	4.52	4.66	4.09	4.14	3.83	3.86
	NS	4.82	4.82	4.07	4.07	3.77	3.77
	DS	5.01	5.07	4.26	4.31	3.95	3.99
λ = -2, r = 4%	*Optimal*	3.78	4.10	3.71	3.83	3.61	3.64
	OV	4.40	4.55	3.93	3.98	3.69	3.71
	ITM	4.40	4.54	3.94	3.99	3.70	3.73
	NS	4.73	4.73	3.92	3.92	3.64	3.64
	DS	4.91	4.97	4.11	4.16	3.82	3.86
λ = 0, r = 2%	*Optimal*	3.14	3.38	3.12	3.23	3.09	3.13
	OV	3.58	3.69	3.32	3.37	3.19	3.20
	ITM	3.59	3.69	3.32	3.37	3.19	3.22
	NS	3.79	3.79	3.32	3.32	3.14	3.14
	DS	3.97	4.01	3.49	3.53	3.31	3.34
λ = 0, r = 3%	*Optimal*	3.59	3.85	3.54	3.66	3.48	3.50
	OV	4.08	4.20	3.75	3.80	3.56	3.57
	ITM	4.09	4.21	3.76	3.81	3.56	3.59
	NS	4.33	4.33	3.74	3.74	3.50	3.50
	DS	4.51	4.56	3.93	3.97	3.68	3.72
λ = 0, r = 5%	*Optimal*	4.56	4.90	4.47	4.59	4.24	4.25
	OV	5.18	5.33	4.69	4.73	4.31	4.33
	ITM	5.19	5.34	4.69	4.75	4.32	4.35
	NS	5.51	5.51	4.66	4.66	4.25	4.25
	DS	5.71	5.77	4.87	4.93	4.45	4.50

Table 6: Fair guaranteed withdrawal rates x_{WL} in percent under the Heston model for different ratchet mechanisms, policyholder behavior assumptions, market price of volatility risk parameters λ, rates of interest and surrender fees.

The results under the Heston model are very similar to those observed within the Black-Scholes model: Fair withdrawal rates are higher for the product design without ratchet and for lower long-term volatility assumptions. The potential for mispricing is also higher for the product design without any ratchet.

Comparing the results of the Heston model with the corresponding results using the Black-Scholes model shows that the assumption of stochastic equity volatility seems to have only little influence on pricing results for GLWB riders (which is consistent to findings in Kling et al., 2011).

3.3 Quantifying the Risk resulting from Behavioral Assumptions

In a next step, we analyze the loss potential an insurer faces if pricing assumptions for policyholder behavior deviate from actual policyholder behavior.

3.3.1 Results for the Black-Scholes model

Table 7 shows the GLWB rider's value at inception from the insurance company's perspective as a percentage of the single premium if the actual future policyholder behavior as well as equity volatility or interest rates differ from the pricing assumptions. Negative values therefore represent the equivalent of an immediate loss for the insurance company if the insurer charges a certain price for the guarantee that was calculated using assumptions that differ from actual behavior and/or market parameters in a negative way. The results in this table are given for a surrender fee of 3%. We assume that the products are priced assuming an equity volatility in the Black-Scholes model of 22% alongside an interest rate of 4%, and that the actual parameters are either 22% or 25% for the volatility and either 4% or 3% for the rate of interest.

Behavior		Pricing: σ_{BS} = 22%, r = 4% Actual: σ_{BS} = 22%, r = 4%			Pricing: σ_{BS} = 22%, r = 4% Actual: σ_{BS} = 25%, r = 4%			Pricing: σ_{BS} = 22%, r = 4% Actual: σ_{BS} = 22%, r = 3%		
Pricing	Actual	I	II	III	I	II	III	I	II	III
Optimal	*Optimal*	0.0	0.0	0.0	-1.3	-2.2	-2.5	-3.1	-4.0	-4.2
	OV	2.2	1.1	0.7	1.1	-1.2	-1.6	-0.9	-2.8	-3.2
	ITM	2.3	1.1	0.8	1.1	-1.0	-1.4	-0.8	-2.6	-2.9
	NS	4.3	0.7	0.0	3.1	-1.8	-2.5	0.1	-3.7	-4.2
	DS	4.9	2.2	1.8	3.9	0.3	-0.1	1.7	-1.1	-1.4
OV	*Optimal*	-2.2	-1.1	-0.7	-3.7	-3.5	-3.3	-6.1	-5.4	-5.1
	OV	0.0	0.0	0.0	-1.2	-2.3	-2.3	-3.9	-4.2	-4.0
	ITM	0.0	0.1	0.1	-1.2	-2.1	-2.1	-3.6	-3.8	-3.7
	NS	1.3	-0.6	-0.7	0.0	-3.2	-3.3	-3.6	-5.2	-5.1
	DS	2.5	1.3	1.2	1.5	-0.8	-0.8	-1.2	-2.3	-2.1
ITM	*Optimal*	-2.2	-1.2	-0.9	-3.7	-3.6	-3.5	-6.1	-5.6	-5.3
	OV	0.0	-0.1	-0.2	-1.3	-2.5	-2.5	-3.9	-4.3	-4.2
	ITM	0.0	0.0	0.0	-1.3	-2.2	-2.2	-3.6	-3.9	-3.8
	NS	1.3	-0.7	-0.9	0.0	-3.3	-3.5	-3.6	-5.4	-5.3
	DS	2.5	1.2	1.1	1.4	-0.9	-0.9	-1.2	-2.4	-2.2
NS	*Optimal*	-3.2	-0.6	0.0	-4.7	-2.9	-2.5	-7.4	-4.8	-4.3
	OV	-1.0	0.5	0.7	-2.3	-1.8	-1.6	-5.2	-3.6	-3.2
	ITM	-0.9	0.6	0.8	-2.2	-1.6	-1.4	-4.8	-3.2	-2.9
	NS	0.0	0.0	0.0	-1.3	-2.6	-2.5	-5.2	-4.5	-4.3
	DS	1.5	1.7	1.8	0.4	-0.3	-0.2	-2.4	-1.7	-1.4
DS	*Optimal*	-4.7	-2.5	-2.4	-6.3	-5.1	-5.0	-9.5	-7.3	-7.0
	OV	-2.6	-1.5	-1.5	-3.9	-3.9	-3.9	-7.2	-6.0	-5.8
	ITM	-2.4	-1.3	-1.3	-3.7	-3.5	-3.6	-6.6	-5.4	-5.3
	NS	-2.0	-2.2	-2.3	-3.3	-4.9	-5.0	-7.6	-7.2	-7.0
	DS	0.0	0.0	0.0	-1.1	-2.1	-2.1	-4.3	-3.8	-3.5

Table 7: GLWB rider value at inception as percentage of the single premium if actual policyholder behavior and/or parameters in the Black-Scholes model differ from pricing assumptions.

Pricing assumption *DS*

We first look at the case where deterministic surrender probabilities are assumed in the pricing of the contract. Clearly, the potential loss is the highest if policyholders behave optimally. In particular, if assumptions about equity volatility and interest rates are correct, the insurance company’s loss is 4.7% of the single premium paid if no ratchet is included, 2.5% of the single premium paid in the case of the lookback ratchet and 2.4% of the single premium paid for the remaining WBB ratchet. In line with the results from Section 3.2, the loss potential if only the assumption about policyholder behavior is incorrect is significantly lower if ratchets are included into the product and is the lowest for the product design with the most valuable ratchet mechanism. If policyholders in reality do not behave optimally but either surrender according to one of the path-dependent rules (*OV* or *ITM*) or do not surrender at all (*NS*), then the potential loss roughly lies between 1% and 3% of the single premium paid if the assumptions regarding the market parameters are correct. Again, the riskiest product design is the one without ratchet. However, the product designs with ratchets are more sensitive to changes in volatility:

If (additionally to policyholder behavior assumptions) volatility assumptions are also wrong (i.e. σ_{BS} = 25%), the insurance company's loss increases by at least 2% for the products with ratchet, while for the product without ratchet, the increase is between 1.1% and 1.6% of the single premium paid.

If (additionally to policyholder behavior assumptions) interest rate assumptions are also wrong (i.e. r = 3%), the insurance company's loss increases by at least 3.5% of the single premium paid. This increase is rather similar across the three product designs. Losses can now go up to almost 10% of the premium paid if deterministic policyholder behavior is assumed and actual behavior is optimal.

If policyholders in reality do not surrender at all (*NS*), independent of volatility and interest rates, the loss potential is higher than with any path-dependent behavior (*OV* and *ITM*) for the product designs with ratchets and lower for the product design without ratchet. At the same time, not surrendering seems to be very close to the optimal strategy for product III, which has a rich ratchet mechanism. Also, the interest rate sensitivity is the highest if policyholders do not surrender.

Pricing assumption *NS*

We now look at the case where no surrender is assumed in pricing. In this case (if assumed and actual market parameters coincide), for the products with ratchet, the insurer would realize a gain if any of the other non-optimal policyholder behavior patterns occurs, i.e. *OV*, *ITM* or *DS*. Thus, the insurance company can reduce the risk resulting from policyholder behavior by including a strong ratchet mechanism into the product and at the same time assuming no surrender in pricing the contract. A rich ratchet mechanism can prevent high values of the option to surrender under almost all circumstances. This can be a very effective means to manage policyholder behavior risk.

The effect of wrong volatility assumptions on the insurance company's loss is similar to the one observed when deterministic surrender is assumed: The loss increases by at least 2% for the products with ratchet, while for the product without ratchet the increase is between 1.1% and 1.5% of the single premium paid. Similar increases can also be observed for all other pricing assumptions.

The absolute increase caused by wrong interest rate assumptions, however, is less pronounced than if deterministic surrender is assumed in pricing. The increase is still similar for the different product designs.

Pricing assumptions *OV* and *ITM*

A common path-dependent assumption about policyholder behavior suggests that surrender rates are influenced by the in-the-moneyness (as e.g. described in American Academy of Actuaries, 2005) or (more directly) the value of the guarantee. Although more conservative than assuming purely deterministic behavior, these assumptions can still be quite dangerous: If, for instance, a remaining WBB ratchet is in place and policyholders do not surrender at all, the potential loss amounts to almost 1% of the single premium paid even if assumed market parameters are correct. A slightly smaller loss occurs in case of a lookback ratchet, whereas if no ratchet is in place, there even is a profit. However, if policyholders behave optimally, the potential loss, again, is the highest for the product design without ratchet and amounts to 2.2% of the single premium paid.

If additionally actual market parameters deviate from assumed market parameters, potential losses increase up to over 5% of the single premium paid. The structure of the increase is similar to the effects observed previously, only the differences between the different product designs with respect to their interest rate sensitivity are now slightly higher. This also holds for the following optimal behavior assumption.

Pricing assumption *Optimal*

The most conservative assumption about the policyholders' behavior is of course to assume that they follow an optimal surrender strategy. As a result, losses due to mispricing only occur if additionally to policyholder behavior also market parameters are different than assumed. In this case, the profit from the potentially over-conservative behavior assumption is reduced by these losses. Of course, the losses are the highest if actual behavior is either optimal or, in the case of the remaining WBB ratchet, where not surrendering yields very similar results to optimal behavior, if policyholders do not surrender at all.

Summarizing, we find that the product design without ratchet shows the highest sensitivity to deviations from assumed policyholder behavior. On the other hand, it is the design with the least sensitivity to deviations from assumed volatility, while the negative effect of an overestimated level of interest rates is roughly the same for all three product designs.

3.3.2 Results for the Heston model

Table 8 shows similar results to those presented in the previous section but now using the Heston model. For all results in this table, the surrender fee was set to 3% and, for pricing, the market-price-of-risk factor was set to λ=0 and a rate of interest of 4% was assumed.

Behavior		Pricing: λ=0, r = 4%; Actual: λ=0, r = 4%			Pricing: λ=0, r = 4%; Actual: λ=-2, r = 4%			Pricing: λ=0, r = 4%; Actual: λ=0, r = 3%		
Pricing	Actual	I	II	III	I	II	III	I	II	III
	Optimal	0.0	0.0	0.0	-1.2	-2.1	-2.4	-3.1	-3.9	-4.1
	OV	2.0	1.0	0.7	1.0	-1.1	-1.5	-1.0	-2.8	-3.1
Optimal	*ITM*	2.1	1.1	0.8	1.0	-0.9	-1.3	-1.0	-2.5	-2.8
	NS	4.0	0.6	0.0	2.8	-1.7	-2.4	-0.3	-3.7	-4.1
	DS	4.6	2.2	1.8	3.6	0.3	-0.1	1.4	-1.1	-1.3
	Optimal	-2.0	-1.0	-0.7	-3.4	-3.3	-3.1	-5.9	-5.3	-5.0
	OV	0.0	0.0	0.0	-1.2	-2.2	-2.2	-3.8	-4.1	-3.9
OV	*ITM*	0.0	0.1	0.2	-1.1	-1.9	-1.9	-3.5	-3.7	-3.5
	NS	1.1	-0.5	-0.7	-0.1	-3.0	-3.1	-3.8	-5.1	-5.0
	DS	2.4	1.2	1.2	1.3	-0.7	-0.7	-1.3	-2.2	-2.0
	Optimal	-2.0	-1.1	-0.9	-3.4	-3.4	-3.3	-5.9	-5.4	-5.2
	OV	0.0	-0.1	-0.2	-1.2	-2.3	-2.4	-3.9	-4.2	-4.1
ITM	*ITM*	0.0	0.0	0.0	-1.2	-2.0	-2.1	-3.6	-3.8	-3.7
	NS	1.1	-0.7	-0.9	-0.2	-3.1	-3.3	-3.8	-5.2	-5.2
	DS	2.3	1.1	1.1	1.3	-0.8	-0.8	-1.4	-2.3	-2.2
	Optimal	-2.8	-0.5	0.0	-4.3	-2.7	-2.4	-7.1	-4.7	-4.1
	OV	-0.9	0.5	0.7	-2.1	-1.7	-1.5	-5.0	-3.5	-3.1
NS	*ITM*	-0.8	0.5	0.8	-2.0	-1.4	-1.3	-4.6	-3.1	-2.8
	NS	0.0	0.0	0.0	-1.3	-2.4	-2.4	-5.1	-4.4	-4.1
	DS	1.5	1.7	1.8	0.4	-0.2	-0.1	-2.4	-1.7	-1.3
	Optimal	-4.4	-2.5	-2.3	-5.9	-4.8	-4.8	-9.1	-7.1	-6.8
	OV	-2.4	-1.4	-1.5	-3.6	-3.7	-3.8	-7.0	-5.8	-5.6
DS	*ITM*	-2.2	-1.2	-1.2	-3.4	-3.3	-3.4	-6.3	-5.2	-5.1
	NS	-1.9	-2.1	-2.3	-3.2	-4.6	-4.8	-7.5	-7.0	-6.8
	DS	0.0	0.0	0.0	-1.1	-2.0	-2.0	-4.2	-3.7	-3.4

Table 8: GLWB rider value at inception as percentage of the single premium if actual policyholder behavior and/or parameters in the Heston model differ from pricing assumptions.

Again, the results observed under the Heston model are very similar to those observed under the Black-Scholes model. Thus, the potential for mispricing arising from wrong assumptions about policyholder behavior or a wrong level of volatility or interest rates seems to be much higher than the potential loss arising from ignoring the stochasticity of equity volatility. However, by solely calculating the rider value of the guarantee we implicitly assume perfect hedge effectiveness which is not given in reality.

Kling et al. (2011) have shown that the impact of stochastic volatility on hedge efficiency of such products is typically much higher than on pricing. Therefore, we now analyze how this result relates to different assumptions about policyholder behavior.

4 Analysis of Hedge Efficiency

In this section, we analyze the performance of a hedging program an insurer might apply in order to reduce the financial risk – and thus also the required economic capital – resulting from selling GLWB guarantees. We analyze this performance under different assumptions regarding policyholder behavior and particularly analyze the case where the policyholder behavior assumed by the insurer for pricing and hedging differs from the actual behavior of the policyholders.

In what follows, we first describe the analyzed delta-hedging strategy; we then define the risk measures that we use to compare the (simulated) hedge performance, and finally, we present the simulation results in the last part of this section. The methodology we use is similar to the one used by Kling et al. (2011).

4.1 Hedge Portfolio

We assume that an insurer has sold a pool of policies with GLWB guarantees. We denote by Ψ_t the cumulative option value for that pool of guarantees, i.e. the sum of the option values V_t^G of each policy as defined in Section 2.3. We assume that the insurer cannot influence the value of Ψ_t by changing the underlying fund (e.g. changing the fund's exposure to risky assets or forcing the policyholder to switch to a different, e.g. less volatile, fund). We further assume that the insurer invests the guarantee fees as well as surrender fees in a hedge portfolio Π_t^H and applies some hedging strategy within this portfolio. In case the guarantee of a policy is triggered, the guaranteed payments due are deducted from this portfolio. Thus,

$$\Pi_t := -\Psi_t + \Pi_t^H \qquad (12)$$

is the insurer's cumulative profit/loss (in what follows sometimes just denoted as the insurer's profit) at time t stemming from the guarantee and the corresponding hedging strategy. We assume the value of the guarantee to be marked-to-model, where the same model the insurer uses for hedging is used for the valuation of Ψ_t.

For the simulations in the following section, we assume that the insurer uses the Black-Scholes model for hedging purposes and applies a simple delta-hedging strategy within the hedge portfolio Π_t^H: In order to immunize the portfolio against small changes in the underlying's spot price S_t (i.e. to attain delta-neutrality), the quantity of exposure to the underlying within the insurer's hedge portfolio is determined as the delta of Ψ_t, i.e. the partial derivative of Ψ_t with respect to S_t.

We assume that the hedge portfolio is rebalanced on a monthly basis, using central finite differences calculated via Monte-Carlo simulation as approximation for the partial derivative of Ψ_t with respect to S_t.

4.2 Risk Measures

We use the following three measures to compare the different hedging strategies. All measures will be normalized as a percentage of the premium volume at t=0:

- $E_P\left[e^{-rT}\Pi_T\right]$, the expectation of the discounted final value of the insurer's profit under the real-world measure P. This is a measure for the insurer's expected profit and constitutes the "performance" measure in our context. A value of 1 means that, in expectation, for a single premium of 100 paid by the client, the insurance company's discounted profit from selling and hedging the guarantee is 1.

- $CTE_{1-\alpha}(\chi) = E_P\left[-\chi \middle| -\chi \geq VaR_\alpha(\chi)\right]$, the conditional tail expectation of the random variable χ, where $\chi = \min\left\{e^{-rt}\Pi_t \middle| t = 0,1,\ldots,T\right\}$ is defined as the minimum of the discounted values of the insurer's profit/loss at all policy calculation dates and $VaR_\alpha(\chi)$ denotes the Value at Risk of the variable χ at the level α. This is a measure for the insurer's risk resulting from a certain hedging strategy: it can be interpreted as the additional amount of money that would be necessary at outset such that the insurer's portfolio would never become negative over the life of the contract, even if the market develops according to the average of the α (e.g. 10%) worst scenarios in the stochastic model. Thus a value of 1 means that, in expectation over the α worst scenarios, for a single premium of 100 paid by the client, the insurance company would need to hold 1 additional unit of capital upfront.

- $CTE_{1-\alpha}(e^{-rT}\Pi_T) = E_P\left[-e^{-rT}\Pi_T \middle| -e^{-rT}\Pi_T \geq VaR_\alpha(e^{-rT}\Pi_T)\right]$, the conditional tail expectation of the discounted profit/loss' final value. This is also a risk measure which, however, focuses on the value of the profit/loss at time T, i.e. after all liabilities have been met, and does not account for negative portfolio values over time. Thus a value of 1 in the above table means that, in expectation over the α worst scenarios, for a premium of 100 paid by the client, the insurance company's expected loss is 1. By definition, of course, $CTE_{1-\alpha}(\chi) \geq CTE_{1-\alpha}(e^{-rT}\Pi_T)$.

4.3 Simulation Results

In the numerical analyses below, we set α=10% for both risk measures and assume a pool of identical policies with parameters as given in Section 3. We assume that mortality within the population of insured occurs according to the best-estimate probabilities given in the DAV 2004R table. As our analysis focuses on model risk rather than parameter risk, we use the parameters for the capital market models presented in Section 3 for both, the hedging and the data-generating model. The results are calculated using 10,000 Monte-Carlo paths for the simulation, 1,000 paths for each of the valuations used in the calculation of the central finite differences and 10,000 paths for each of the valuations of Ψ_t used to calculate χ.

4.3.1 Results for the Black-Scholes model

Table 9 gives the results for different combinations of behavioral assumptions made by the insurer and actual behavior within the pool of policies. The Black-Scholes model with σ_{BS}=22% and r=4% is thereby used as the hedging model of the insurer and as a model of the real-world progression of the capital market (with μ=7%).

Behavior		Product I			Product II			Product III		
Pricing / Hedging	Actual	$E_P[e^{-rT}\Pi_T]$	$CTE[\chi]$	$CTE[e^{-rT}\Pi_T]$	$E_P[e^{-rT}\Pi_T]$	$CTE[\chi]$	$CTE[e^{-rT}\Pi_T]$	$E_P[e^{-rT}\Pi_T]$	$CTE[\chi]$	$CTE[e^{-rT}\Pi_T]$
Optimal	*Optimal*	-0.1	2.4	2.2	0.1	3.1	2.8	0.0	3.2	2.8
	OV	3.4	1.8	0.8	1.7	2.5	1.5	0.7	2.8	2.0
	ITM	3.5	1.9	0.9	1.7	2.3	1.4	0.8	2.3	1.6
	NS	8.7	2.5	1.8	1.9	3.1	2.3	0.1	3.4	2.8
	DS	7.8	1.8	0.9	2.9	2.1	0.9	1.8	2.0	0.9
OV	*Optimal*	-2.7	7.5	7.5	-1.3	4.5	4.5	-0.7	3.4	3.3
	OV	-0.1	1.6	1.4	0.0	2.7	2.3	0.0	2.6	2.2
	ITM	0.0	1.5	1.4	0.1	2.4	2.0	0.2	2.2	1.9
	NS	2.9	2.2	2.1	-0.3	3.2	2.9	-0.7	3.3	3.1
	DS	3.2	0.9	0.5	1.2	2.0	1.1	1.2	1.6	0.9
ITM	*Optimal*	-2.8	7.5	7.5	-1.4	4.8	4.8	-0.9	3.7	3.6
	OV	-0.1	1.7	1.5	-0.1	2.9	2.6	-0.2	2.9	2.5
	ITM	0.0	1.6	1.4	0.0	2.5	2.3	0.0	2.4	2.1
	NS	2.9	2.3	2.2	-0.4	3.3	3.1	-0.9	3.6	3.4
	DS	3.2	1.0	0.5	1.1	2.1	1.3	1.0	1.9	1.1
NS	*Optimal*	-4.5	15.4	15.4	-1.0	5.1	5.0	-0.1	3.0	2.8
	OV	-2.1	11.0	11.0	0.3	3.0	2.7	0.6	2.2	1.7
	ITM	-2.0	10.8	10.8	0.3	2.6	2.3	0.8	1.9	1.4
	NS	0.0	1.3	1.2	0.0	3.0	2.6	0.0	2.7	2.4
	DS	0.8	3.9	3.8	1.4	2.1	1.3	1.7	1.4	0.7
DS	*Optimal*	-5.5	13.3	13.3	-2.6	5.9	5.9	-2.3	5.1	5.1
	OV	-2.9	8.0	8.0	-1.4	4.0	4.0	-1.4	3.9	3.8
	ITM	-2.7	7.6	7.6	-1.1	3.6	3.6	-1.1	3.4	3.4
	NS	-1.2	5.3	5.3	-1.9	5.0	5.0	-2.2	5.0	5.0
	DS	0.0	1.2	1.0	0.0	2.5	2.2	0.0	2.3	2.0

Table 9: Hedge efficiency results using the Black-Scholes model as data-generating model.

Pricing assumption *DS*

We first look at the case where the insurer assumes deterministic surrender in pricing and hedging the contract. If, in reality, the pool of policyholders behaves exactly according to the same pattern, the insurer's expected profit is close to zero for all different product designs. (Note that we assume that the insurer priced the contracts without incorporating any profit margin.) On average over the 10% worst scenarios, the present value of the insurer's final loss averages to 1.0% of the single premium for product design I. The corresponding values are 2.2% and 2.0% for product designs II and III, respectively. The CTE of the present value of the maximum loss over all policy calculation dates is slightly higher. Similar results are observed for other assumptions about the policyholder behavior as long as assumed policyholder behavior and realized policyholder behavior coincide.

If the insurer assumes deterministic surrender but policyholders actually behave according to the considered function of the in-the-moneyness (*ITM*) or the considered function of the option value (*OV*), the insurer's expected loss significantly increases. The expected loss for the product design without ratchet is roughly 3% of the single premium paid and hence more than twice as high than that for the product designs with ratchet. In the case of optimal surrender, the expected loss further increases to 5.5% of the single premium paid in the case without ratchet and about half that value for the products with ratchet. With the expected loss, also the risk increases. Assuming deterministic surrender for product I results in a CTE of final losses between 7.6% and 13.3% of the single premium paid if actual policyholder behavior is path-dependent or even optimal. This risk is reduced by roughly 50% by including ratchets into the product design. If policyholders do not surrender at all (*NS*), the risk is almost the same for all product designs. Consistent with the pricing results above, for product design III, the results for no surrender and optimal policyholder behavior are almost identical since optimal surrender for this product is close to no surrender. For product design I, the risk if policyholders do not surrender is lower than for any path-dependent behavior.

Pricing assumption *NS*

If no surrender is assumed in pricing and hedging, the results are rather diverse. Actual deterministic policyholder behavior leads to a positive expected profit for all product designs and rather limited risk for product designs II and III. However, risk measures for product design I are almost 4%. If policyholders actually behave according to the considered function of the in-the-moneyness (*ITM*) or the considered function of the option value (*OV*), the insurer's expected profit is slightly positive for product designs II and III and around -2% for product design I. Interestingly, while for the product designs with ratchet, the risk also is rather limited, the risk measures for product design I exceed 10%. Thus, if no ratchet is included in the product design, the assumption of no surrender is rather risky. If policyholder behavior is optimal, the risk for this product even increases to 15%.

Pricing assumptions *OV* and *ITM*

If policyholder behavior is assumed to occur according to the considered function of the in-the-moneyness (*ITM*) or the considered function of the option value (*OV*) and ratchets are included into the product design (products II and III), the expected loss for the insurer (even in the case of optimal policyholder behavior) is below 1.5% of the single premium paid. Furthermore, the considered risk measures remain below 5%. Again, for product design III, actual policyholder behavior without surrender turns out to be almost as risky as optimal policyholder behavior. For product design I, however, no surrender leads to expected profits of 2.9% of the single premium paid and risk measures below 2.3% while optimal behavior leads to a risk of 7.5% and a negative expected profit. Deterministic behavior under both assumptions and for all product designs leads to expected profits and rather low risk.

Pricing assumption *Optimal*

Not very surprisingly, the most conservative assumption of optimal policyholder behavior always leads to the highest expected profit. If actual behavior is deterministic or no surrender, for product design I the expected profit reaches 7.8% and 8.7%, respectively. Also, risk is rather limited and for all product designs below 3.4%. However, it is worth noting that the risk if policyholders do not surrender for product design III (3.4%) is slightly higher than in the case of optimal surrender. We attribute this to the fact that in the case of optimal surrender, all policyholders surrender at the same time. Thus, hedging is needed for a potentially shorter period of time, resulting in a reduced hedging error. Also, for product III, assuming optimal surrender results in a slightly higher risk than if no surrender is assumed. We attribute this to a more stable hedging in the case of no surrender (as there are no decisions whether all of the policyholders either stay or leave) and some imperfections in the Longstaff-Schwartz algorithm we used.

4.3.2 Results for the Heston model

Table 10 shows the same results as in Table 9, but now using the Heston model as data-generating model instead of the Black-Scholes model, with parameters as stated in Table 4 and with μ=7%.

Behavior		Product I			Product II			Product III		
Pricing / Hedging	Actual	$\mathrm{E}_P[e^{-rT}\Pi_T]$	$\mathrm{CTE}[\chi]$	$\mathrm{CTE}[e^{-rT}\Pi_T]$	$\mathrm{E}_P[e^{-rT}\Pi_T]$	$\mathrm{CTE}[\chi]$	$\mathrm{CTE}[e^{-rT}\Pi_T]$	$\mathrm{E}_P[e^{-rT}\Pi_T]$	$\mathrm{CTE}[\chi]$	$\mathrm{CTE}[e^{-rT}\Pi_T]$
Optimal	*Optimal*	0.1	3.3	2.9	0.3	4.1	3.7	0.0	4.5	4.0
	OV	3.4	2.5	1.5	1.9	3.5	2.5	0.7	3.9	3.1
	ITM	3.5	2.7	1.6	1.9	3.4	2.4	0.8	3.5	2.7
	NS	8.7	3.5	2.8	2.2	4.3	3.6	0.1	4.6	4.0
	DS	7.8	2.4	1.1	3.1	3.0	1.4	1.7	2.9	1.7
OV	*Optimal*	-2.4	6.4	6.4	-1.0	5.0	4.9	-0.7	4.9	4.7
	OV	0.1	2.5	2.2	0.3	3.9	3.4	0.0	4.1	3.6
	ITM	0.1	2.4	2.2	0.3	3.6	3.2	0.2	3.7	3.3
	NS	2.9	3.4	3.3	0.0	4.7	4.4	-0.7	4.9	4.7
	DS	3.2	1.7	0.8	1.4	3.1	2.0	1.1	3.0	2.0
ITM	*Optimal*	-2.5	6.4	6.4	-1.1	5.1	5.1	-0.9	5.1	4.9
	OV	0.1	2.5	2.2	0.1	4.0	3.6	-0.2	4.3	3.8
	ITM	0.1	2.5	2.2	0.2	3.7	3.3	0.0	3.8	3.4
	NS	2.9	3.5	3.3	-0.1	4.8	4.5	-0.9	5.1	4.9
	DS	3.2	1.7	0.8	1.3	3.2	2.1	1.0	3.1	2.2
NS	*Optimal*	-4.2	13.4	13.4	-0.7	4.8	4.6	-0.1	4.3	3.9
	OV	-1.9	9.3	9.3	0.5	3.7	3.1	0.6	3.7	2.9
	ITM	-1.8	9.2	9.2	0.6	3.5	2.9	0.7	3.3	2.6
	NS	0.0	2.3	2.0	0.3	4.2	3.7	0.0	4.3	3.8
	DS	0.8	3.3	3.2	1.6	3.0	1.8	1.6	2.7	1.5
DS	*Optimal*	-5.2	11.7	11.7	-2.3	6.8	6.8	-2.3	6.7	6.7
	OV	-2.8	6.8	6.8	-1.1	5.4	5.3	-1.4	5.5	5.4
	ITM	-2.6	6.6	6.6	-0.9	5.0	5.0	-1.1	5.0	4.9
	NS	-1.1	6.4	6.4	-1.5	6.7	6.6	-2.3	6.7	6.7
	DS	0.0	2.0	1.8	0.3	3.6	3.2	0.0	3.7	3.3

Table 10: Hedge efficiency results using the Heston model as data-generating model.

Changing the data-generating model from Black-Scholes to Heston does not have any substantial impact on the expected profit, independent of the product design and the assumed policyholder behavior. Also, the structure of the results, i.e. the relation between the results for the different products and the different client behavior patterns is very similar. However, the absolute values of the risk measures change. While product design I appears to be less risky in case of path-dependent behavior under the Heston model if deterministic or no surrender is assumed, the risk for product designs II and III increases. The results show that product designs II and III display a higher sensitivity to volatility than the design without ratchet (I). This is in line with the results of Section 3.3.

We can conclude that assumptions about policyholder behavior can bear significant risk for the insurer, especially if such assumptions are too aggressive, i.e. if policyholders' behavior is closer to optimal behavior than assumed. However, this risk can be significantly reduced by means of product design and making appropriate behavioral assumptions. The latter, however, also increases the price of the product and may result in a lower competitiveness of the product. While the product designs with ratchet features (II and III) appear to be less sensitive to policyholder behavior, our results indicate that they may be harder to hedge and are more sensitive to changes in volatility and/or model risk, respectively.

5 Conclusions

In the present paper, we have analyzed the impact of policyholder behavior on pricing, hedging and hedge efficiency of different GLWB guarantees in variable annuities. We have considered several types of policyholder behavior ranging from deterministic surrender over path-dependent surrender to optimal strategies. We have found that the price of the guarantee strongly depends on the assumed policyholder behavior and there is a significant potential for mispricing if actual policyholder behavior deviates from assumed behavior. Comparing products with different ratchet mechanisms, we find that this potential for mispricing is the smallest for the product design with the most valuable ratchet mechanism.

Analyses of an insurer's hedging strategy showed that both, the insurer's expected profit and the insurer's risk (quantified by CTE measures), depend heavily on the deviation between assumed and actual policyholder behavior as well as the chosen product design. We find that the product design without ratchet shows the highest sensitivity to changes in policyholder behavior. On the other hand, it is the design with the least sensitivity to changes in volatility and the potentially easiest one to hedge. We also find that the impact of stochastic volatility on hedging (and the insurer's risk) is much higher than on pricing (and the insurer's expected profit).

In future research, it would be interesting to combine the analyses of model risk performed in Kling et al. (2011) with the analyses of policyholder behavior risk and quantify how the insurer's risk depends on a simultaneous deviation from reality of assumptions regarding policyholder behavior and the capital market model. It might also be worthwhile to analyze different types of variable annuity guarantees and see whether different types of guarantees (e.g. GMAB or GMIB) display higher or lower behavioral risk than the GLWB designs considered in this paper.

Our analyses so far have been performed on the level of an individual policy. Since hedging errors are not necessarily additive over a pool of policies, it would be worthwhile to analyze how the results with respect to risk management and hedge efficiency change for a heterogeneous pool of policies.

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