VARIABLE ANNUITIES WITH GUARANTEED LIFETIME WITHDRAWAL BENEFITS: AN ANALYSIS OF RISK-BASED CAPITAL REQUIREMENTS

BY

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ABSTRACT

Under risk-based regulatory regimes like Solvency II in the EU, the risk profile of a variable annuity directly affects the amount of capital that providers are required to hold. Therefore, providers of variable annuities not only face the challenge to hedge against changes in the value of embedded guarantees, but are also exposed to potential additional capital needs due to changes in their capital requirements. Both, the value of embedded guarantees as well as corresponding capital requirements, are dependent on market parameters and, thus, subject to changes.

We analyze the risk profile of a pool of variable annuity policies with Guaranteed Lifetime Withdrawal Benefit (GLWB) riders with regard to the pool's key financial risk drivers: equity returns, implied equity volatility and interest rates. In a simulation study, we analyze the effectiveness of different stylized hedging programs over a one-year time horizon and compute indicators for risk-based capital requirements. The approach we use is comparable to an internal model type approach under Solvency II. We also analyze the impact changing market environments have on risk profile, hedge effectiveness and capital requirements, similar to a forward-looking analysis in the context of the mandatory Own Risk and Solvency Assessment (ORSA) under Solvency II.

We find that, in addition to the stress from potentially unhedged increases in the value of liabilities, changes in the market environment can have a substantial impact on capital requirements. As a result, GLWB providers face the risk of increases in their risk-based capital requirements and, thus, the need for capital injections – even without pricing errors or malfunctioning of the hedging program. However, there are also cases where an increase in the value of liabilities is accompanied by a decrease of capital requirements, reducing the overall impact on the provider.

KEYWORDS

Variable Annuity, Guaranteed Lifetime Withdrawal Benefits (GLWB), Hedge Performance, Risk-Based Capital Requirements, Stochastic Interest Rates, Stochastic Equity Volatility

1 INTRODUCTION

Variable annuities with Guaranteed Lifetime Withdrawal Benefit (GLWB) riders provide policyholders an opportunity to combine fund investments with a protection against outliving their retirement savings. While the savings remain invested in funds, the GLWB gives them the right to lifelong regular withdrawals from their account, even after the invested amount is depleted and the account balance dropped to zero. Usually, variable annuity providers receive ongoing guarantee charges deducted from the policyholder's account in compensation for this guarantee. In other words, this type of variable annuity embeds a variant of ruin-contingent life annuity (cf. Huang et al., 2014), where the guarantee provider starts to pay a lifelong annuity as soon as the account value (reduced by pre-defined withdrawals) hits zero. Modern GLWB riders typically also include a form of ratchet mechanism, through which the guaranteed withdrawal amount may increase during the lifetime of the contract if the underlying fund performs well. To counter the financial risks that come with this guarantee, insurers and other providers typically implement hedging programs, which aim to mitigate the influence market movements and changing market conditions have on the provider's profit and loss (P&L).

Hedging programs can be quite effective in mitigating the financial risks inherent in GLWB riders, but they usually do not allow for a perfect replication of the changes in the value of liabilities, due to discrete rebalancing and other imperfections (cf. Ledlie et al., 2008). Hence, the provider's P&L with regard to its GLWB business remains subject to fluctuations, even if a hedging program is implemented. Since it is also not feasible to control for every single influencing market parameter, hedging programs normally only aim to control the influence of key risk drivers like the underlying fund's return, interest rate levels and implied equity volatility up to a certain degree. Additionally, policyholder behavior can have a substantial impact on the variable annuity provider's P&L (cf. Kling et al., 2014). As a consequence, variable annuity providers face considerable risks and, under risk-based regulatory regimes like Solvency II, need to equip their variable annuity business with adequate financial resources in order to meet capital requirements.

In the context of GLWB, while the analysis of the impact of key financial risk drivers on the value of liabilities is rather straightforward, it is not entirely clear to which extent this impact transfers to the magnitude of the inherent risk and, thus, to the risk-based capital requirements stipulated by a regulator. To our knowledge, this has not yet been analyzed in the scientific literature, which is why this paper aims to fill this gap by analyzing and illustrating the risk profile and resulting risk-based capital requirements in the context of pools of variable annuity contracts with GLWB riders. The approach we use is comparable to an internal model type approach under Solvency II and the analyses of the impact changing market environments have on the risk profile, hedge effectiveness and capital requirements are comparable to forward-looking analyses in the context of the mandatory Own Risk and Solvency Assessment (ORSA) under Solvency II.

The pricing and valuation of variable annuity contracts with guarantees have been studied in great detail, with Milvesky & Salisbury, 2006, being the first to analyze the valuation of guaranteed minimum withdrawal benefits, and with Bauer et al., 2008, as well as Bacinello et al., 2011, providing general frameworks for the valuation of variable annuities with all types of guarantees. Regarding the risk management of variable annuities, Cathcart et al., 2015 provide schemes to efficiently calculate the "Greeks" of a variable annuity liability via Monte Carlo simulation, while Kling et al., 2011, show that it is important to include stochastic volatility in the modeling when assessing the risk inherent in variable annuity contracts. The same authors analyze the impact of policyholder behavior on the valuation and risk assessment of GLWB (cf. Kling et al., 2014). Forsyth & Vetzal, 2014, present an optimal stochastic control framework, in which they analyze the sensitivity of the cost of hedging a variable annuity with GLWB to various economic and contractual assumptions. The hedging costs for variable annuities with combined guaranteed lifelong withdrawal and death benefits (GLWDB) is analyzed in Azimzadeh et al., 2014, in which the authors also argue that, when analyzing dynamic policyholder behavior from an insurer's perspective, it is better to use the term "loss-maximizing strategy" instead of "optimal strategy".

The paper is organized as follows. In Section 2, we describe the model framework we use for our analysis, including the product design of the considered GLWB (Section 2.1), the modeling of the pool of policies (Section 2.2), the used market models (Section 2.3) and the modeling of the stylized hedging programs implemented by the insurer (Section 2.4). Our numerical results are presented in Section 3, where we first present the parameters and assumptions used in the numerical modeling of the pool of policies in Section 3.1 and present the results of our analysis regarding the risk profile of the modeled pool of policies and its sensitivity to different financial risk factors in Section 3.2. The results of our simulation study are presented in Sections 3.3 and 3.4, respectively, where we analyze the effectiveness of the stylized hedging programs and, based on the distribution of the provider's resulting P&L, compute different indicators for risk-based capital requirements. Finally, Section 4 concludes.

2 MODEL FRAMEWORK

In this section, we describe the model used in our analysis of the risk profile of a pool of variable annuities with GLWB riders. Mainly, we follow the modeling approach used in Bacinello et al., 2011, and Bauer et al., 2008. We consider only two product designs of the GLWB rider: a GLWB rider with a ratchet mechanism, i.e. a product design where increases in the guaranteed withdrawal amount are possible and are regularly checked for, as well as a GLWB rider without ratchet mechanism. We use the product design without ratchet in order to illustrate and extract the effect the considered ratchet mechanism of the first product has on the results. We refer the interest reader to Kling et al., 2011, and Kling et al., 2014, for a broader analysis of different designs of the GLWB rider.

Please note that, although we set surrender rates to zero in our computations in Section 3, we still give formulas for surrender in this section – for the sake of model completeness and in order to set the basis for potential subsequent analyses, where surrender might be considered. The reason we do not consider surrender in our computations is that we want to purely focus on the effect financial risk drivers have on risk-based capital requirements. Depending on the approach used to model dynamic policyholder behavior, results may show effects that are not easily interpretable and that are highly dependent on the specific parametrization of the used behavior model (for instance a model where surrender rates are modeled as a function of the "moneyness" of the contract). To consider this in our analyses would be out of the scope of this paper. However, we refer the interested reader to Forsyth & Vetzal, 2014, and Kling et al., 2014, for an analysis of surrender risk, especially due to dynamic and "loss-maximizing" policyholder behavior.

2.1 VARIABLE ANNUITY CONTRACT

We consider a sequence of withdrawal (or "calculation") dates, represented by the points in time $(t_i)_{i=0}^{N}$, where $t_i \in \mathbb{R}_{\geq 0}$ and $t_0 = 0$ represents the inception of the contract. While the number of such calculation dates is not limited by means of the contract, it is limited by (the assumption of) a limiting age, i.e. an age after which survival is deemed impossible.

Let $\mathcal{T} := \{t_i |\ i = 0, \dots, N\}$ denote the set of all calculation dates. On a calculation date after the inception of the contract, a potential ratchet mechanism is applied and the policyholder is allowed to withdraw money up to a specified amount (the guaranteed withdrawal benefit) from the account without affecting the guarantee of the contract. If the withdrawn amount exceeds the guaranteed withdrawal benefit, this is interpreted as partial surrender, which usually reduces the guarantee. Also, on a calculation date of the contract, a potential surrender of the contract is settled (by payout of the surrender benefit) and, in case the insured person has died, a death benefit is paid out to the beneficiaries.

The value of the policyholder's account at time t is denoted by F_t, where F_0 represents the initial investment amount after deduction of acquisition and other upfront charges, i.e. F_0 is the amount that is actually invested in the fund at inception of the contract. The amount of upfront charges (such as acquisition charges) is not relevant in our analysis and therefore not explicitly denoted in the formulas.

For any calculation date t_{i-1}, the account value F_{t_i} at the following calculation date t_i (while the contract is still in force) is calculated as follows

$$F_{t_i} := \max\left(0, \left(F_{t_{i-1}} - B^W_{t_{i-1}}\right) \cdot \frac{S_{t_i}}{S_{t_{i-1}}} \cdot e^{-(\eta^{mc} + \eta^g)\cdot(t_i - t_{i-1})}\right),$$

where

- $B^W_{t_{i-1}}$ denotes the withdrawal amount paid out to the policyholder at time t_{i-1},
- $S_{t_{i-1}}$ and S_{t_i} denote the share prices of the variable annuity's underlying fund at time t_{i-1} or t_i, respectively,
- η^{mc} denotes the continuously deducted management charge as a percentage of the account value, and
- η^g denotes the continuously deducted guarantee charge as a percentage of the account value.

We denote the guaranteed withdrawal benefit at time t by $B^{W,g}_t$ and, in the case of the product design with ratchet, define it as

$$B^{W,g}_t := \begin{cases} \omega \cdot wbb_t, \text{ if } t \in \mathcal{T}, t > 0 \\ 0, \text{ else} \end{cases},$$

where

- ω is the guaranteed withdrawal rate stipulated at inception, and
- wbb_t denotes the "withdrawal benefit base", which, at a calculation date $t \in \mathcal{T}, t > 0$, is defined as

$$wbb_t := \max_{s \in T, s \le t, F_s > 0} \left\{ F_s + \sum_{u \in \mathcal{T}, u < s} B^W_u \right\}.$$

The withdrawal benefit base wbb_t implements the ratchet mechanism of the GLWB rider, i.e. the potential for increases of the guaranteed withdrawal amount if the fund performs well.

In words, the considered ratchet mechanism works as follows: At each calculation date where the account value is still positive, the then-current account value and the sum of withdrawals are added; The withdrawal benefit base then is defined as the highest of these values in the time span between inception and the current calculation date. Thus, the withdrawal benefit base is non-decreasing over time and can only increase as long as the account value is positive – after the account value has fallen to zero no more increases of wbb_t are possible, i.e. the guaranteed withdrawal amount remains the same until the insured person's death. As a consequence, the guaranteed withdrawal amount $B_t^{W,g}$ can only increase and never decrease (not considering the effects of partial surrenders on the guarantee).

Note: an alternative product design to using ongoing guarantee charges that are proportional to the account value (i.e. η^g) are ongoing charges that are proportional to the withdrawal benefit base (e.g. 1% of the wbb_t, annually), but are still deducted from the account value. This way, the guarantee charges received by the insurer are non-decreasing (until the guarantee triggers) and only increase when a ratchet occurs.

With regard to the fund performance needed to trigger an increase of wbb_t, in this product design, the fund performance only has to compensate for the charges of the variable annuity contract, but not for the withdrawals itself (as long as they remain within the limit of the guaranteed amount $B_t^{W,g}$, which we assume to be the case in our analysis).

In our analysis of the characteristics of a pool of GLWB policies, we also consider an alternative product design without a ratchet, in order to illustrate the effect of the considered ratchet mechanism. In this case, the withdrawal benefit base remains constant over the lifespan of the contract and takes the value of the initial investment:

$$\widehat{wbb_t} := F_0, \qquad t \in \mathcal{T}, t > 0.$$

All other formulas remain the same for this alternative product design without ratchet.

With both product designs, we assume the policyholder to always withdraw exactly the guaranteed withdrawal benefit, i.e. the highest amount that can be withdrawn without negatively affecting the guarantee (in typical product designs the guarantee is reduced proportionally if the withdrawal amount exceeds the guaranteed withdrawal benefit). That is, we set $B_t^W = B_t^{W,g}$ for all $t \in \mathcal{T}, t > 0$ in our analysis.

The considered variable annuity contract (irrespective of the ratchet mechanism) does not include a guaranteed minimum death benefit, but rather pays to the beneficiaries the remaining account value at the calculation date following the insured's death. Therefore, the death benefit B_t^D at time $t \in \mathcal{T}, t > 0$ is defined as

$$B_t^D := F_t.$$

If the policyholder decides to surrender (i.e. "cash out") the contract, surrender charges apply. The portion of the account value corresponding to the guaranteed withdrawal amount at that calculation date is not subject to surrender charges. The amount that exceeds the guaranteed withdrawal benefit, however, is reduced by a proportional surrender charge η^S. Therefore, the surrender benefit at time $t \in \mathcal{T}, t > 0$, denoted by B_t^S, is given by

$$B_t^S := F_t - \max\left(0, \left(F_t - B_t^{W,g}\right) \cdot \eta^S\right).$$

2.2 Pool of policies

For our analyses on a portfolio level, we assume a pool of policies with identical contract parameters with regard to inception date, underlying fund, age as well as gender of insured, guarantee, charges, etc. We assume the pool of insured to be homogeneous and large enough to apply the law of large numbers such that mortality henceforth is only expressed as percentage of the pool of insured.

We denote the total number of active contracts within the pool at time t by π_t^A.

Let $q_i \coloneqq q(t_{i-1}, t_i)$ represent the percentage of the insured who are alive at time t_{i-1} and die within the time interval $]t_{i-1}, t_i]$. The total number of contracts where the insured person dies within the time interval $]t_{i-1}, t_i]$, denoted by $\pi_{t_i}^D$, then is given by

$$\pi_{t_i}^D = \pi_{t_{i-1}}^A \cdot q_i.$$

Similarly, let $s_i \coloneqq s\,(t_{i-1}, t_i)$ represent the fraction of policyholders who want to surrender their contracts at the end of the time interval $]t_{i-1}, t_i]$. The total number of policyholders that surrender their contracts at time t_i, denoted by $\pi_{t_i}^S$, then is given by

$$\pi_{t_i}^S = \left(\pi_{t_{i-1}}^A - \pi_{t_i}^D\right) \cdot s_i.$$

Immediately after the calculation date t_i, i.e. after contracts that were surrendered or that ended due to the death of the insured have left the pool of policies, the number of active contracts is given by

$$\pi_{t_i}^A = \pi_{t_{i-1}}^A - \pi_{t_i}^D - \pi_{t_i}^S.$$

Cash flow

From the viewpoint of the insurer, the cash flows from and to the pool of policies with regard to the guarantee (the GLWB rider) are as follows.

In order to define the guarantee charges received by the insurer at a calculation date t_i, we introduce an auxiliary variable that defines the hypothetical account value after fund performance, but without deduction of charges, denoted by $\hat{F}_{t_i}$ and given by

$$\hat{F}_{t_i} \coloneqq \max\left(0, \left(F_{t_{i-1}} - B_{t_{i-1}}^W\right) \cdot \frac{S_{t_i}}{S_{t_{i-1}}}\right).$$

Surrender charges are treated as a "contribution" to financing the guarantee, and therefore are counted towards the guarantee charges received by the insurer. At a calculation date t_i, the guarantee charges $G_{t_i}^C$ received by the insurer from the pool of policies are then given by

$$G_{t_i}^C = \pi_{t_i}^S \cdot \left(F_{t_i} - B_{t_i}^S\right) + \pi_{t_i-1}^A \cdot \left(\frac{\eta^g}{\eta^{mc} + \eta^g} \cdot \left(\hat{F}_{t_i} - F_{t_i}\right)\right).$$

In return, the insurer has to continue the guaranteed withdrawal payments for the lifetime of the insured person after the account value has fallen to zero (i.e. the guarantee has triggered). At a calculation date t_i, the guarantee payment $G_{t_i}^P$ to be made by the insurer with regard to the pool of policies is given by

$$G_{t_i}^P = \pi_{t_i}^A \cdot \max\left(0, B_{t_i}^{W,g} - F_{t_i}\right).$$

2.3 Market model, valuation and real-world projection

Market model

For our simulation, we need to project the price dynamics of the following assets: the price of one share of the variable annuity's underlying fund, denoted by S_t, the price of a risk-free (with regard to default) zero-coupon bond with time to maturity of τ, denoted by $Z_t(\tau)$, the price of the cash account, denoted by C_t, and the prices of simple put options on the underlying fund, denoted by $O_t^P(\tau, K)$, where τ is the time to maturity and K the strike level of the respective option.

For the valuation of these assets, we use the same system of stochastic differential equations as in Cathcart et al., 2015, in which Heston's stochastic volatility model (Heston, 1993) is used for the equity process and combined with stochastic interest rates via the Cox-Ingersoll-Ross model ("CIR", Cox et al., 1985). Therefore, the dynamics of the market's state variables under the risk-neutral measure Q are given by

$$\begin{aligned} dV_t &= \kappa_V^Q\left(\theta_V^Q - V_t\right)dt + \sigma_V^Q\sqrt{V_t}dW_t^{Q,V}, \\ dr_t &= \kappa_r^Q\left(\theta_r^Q - r_t\right)dt + \sigma_r^Q\sqrt{r_t}dW_t^{Q,r}, \\ dS_t &= r_t S_t dt + \sqrt{V_t} S_t dW_t^{Q,S} \\ dC_t &= r_t C_t dt \end{aligned}$$

where $W_t^{Q,V}, W_t^{Q,r}$ and $W_t^{Q,S}$ are Wiener processes under the risk-neutral measure Q.

We assume the Wiener process $W_t^{Q,r}$ of the interest rate process to be independent of the two equity processes. The correlation between $W_t^{Q,V}$ and $W_t^{Q,S}$ is denoted by the correlation factor $\rho_{S,V}$.

The prices of the bond and the equity option are then given by the following expectations

$$\begin{aligned} Z_t(\tau) &= \mathbb{E}_Q\left[\frac{C_t}{C_{t+\tau}}\middle|\, t\right], \\ O_t^P(\tau, K) &= \mathbb{E}_Q\left[\frac{C_t}{C_{t+\tau}}\max(0, K - S_{t+\tau})\middle|\, t\right]. \end{aligned}$$

Within our simulation, the bond price $Z_t(\tau)$ is calculated via the formulas given in Cox et al., 1985. For the Heston stochastic volatility model, Heston, 1993, found a semi-analytical solution for pricing European call and put options using Fourier inversion techniques. In our analyses, we use the numerical scheme proposed in Kahl & Jäckel, 2005, with the approximation that the interest rates are assumed to be deterministic when pricing the put option used for hedging.

Valuation

In order to determine the value of liabilities, we assume a market-consistent valuation of the guarantee within the considered pool of policies. We define this value of liabilities at time t, denoted by V_t^π, as the difference between the expected present value of future guarantee payments made by the insurer and the expected present value of future guarantee charges to be received by the insurer, i.e.

$$V_t^\pi := \mathbb{E}_Q\left[\sum_{s\in\mathcal{T}, s>t} \frac{C_t}{C_s}\left(G_s^P - G_s^C\right)\middle|\, t\right].$$

In our simulation, this expectation is approximated via Monte Carlo estimation, using the QE scheme developed by Andersen, 2008. See Chan & Joshi, 2013, for a further development of this scheme, allowing for long-stepping intervals. However, as we consider weekly rebalancing of the hedge portfolio, this would be not beneficial for our simulation.

Real-world projection of the market

In our simulation, we project the state of the market over time under the real-world measure P. The state of the market at time t is given by the set of state variables (V_t, r_t, S_t, C_t). For this set of state variables, we assume the dynamics under the real-world measure P to be

$$\begin{aligned} dV_t &= \kappa_V^P(\theta_V^P - V_t)dt + \sigma_V^P\sqrt{V_t}dW_t^{P,V}, \\ dr_t &= \kappa_r^P(\theta_r^P - r_t)dt + \sigma_r^P\sqrt{r_t}dW_t^{P,r}, \\ dS_t &= (r_t + \mu)S_t dt + \sqrt{V_t}S_t dW_t^{P,S} \\ dC_t &= r_t C_t dt \end{aligned}$$

where $W_t^{P,V}, W_t^{P,r}$ and $W_t^{P,S}$ are Wiener processes under P.

The correlation structure between the Wiener processes is assumed to be identical under both measures.

2.4 HEDGING

We assume the insurer to have implemented a dynamic hedging program in order to mitigate the financial risk resulting from the GLWB rider within the pool of policies. The hedge portfolio is assumed to consist of four instruments that are regularly reallocated: units of the underlying fund, put options on the underlying fund, zero-coupon paying bonds, and a cash position. Each position may be long (positive weight) or short (negative weight). The respective weights are determined according to the Greeks of the pool of policies, i.e. derivatives of V_t^{π} with regard to different variables.

Note that this means we ignore basis risk, i.e. the variable annuity's underlying fund and the equity indices used within the hedging program are assumed to be the same or to be at least perfectly correlated.

At each hedging (or rebalancing) date, existing bonds in the portfolio are sold and replaced with zero-coupon paying bonds that have a time to maturity of d_Z. Similarly, put options in the portfolio are sold and replaced with "new" put options that have a strike price of 100% of the then-current spot price (i.e. $K_t = S_t$) and a time to maturity of d_O.

The hedge portfolio's value at time t is denoted by Ψ_t . The inflows into the hedge portfolio are given by the guarantee charges paid by the pool of policies, G_t^C, and its outflows by the guarantee payments G_t^P. We assume the hedge portfolio's value to be zero at inception, i.e. we set $\Psi_0 = 0$.

The weights of the hedge portfolio are calculated as follows, where λ_t^S denotes the number of fund shares, λ_t^Z the number of zero-coupon bonds, λ_t^O the number of options and λ_t^C the amount that is invested in the cash account:

$$\lambda_t^O = \frac{\frac{\partial V_t^{\pi}}{\partial V_t}}{\frac{\partial O_t^P(d_O, K_t)}{\partial V_t}},$$

$$\lambda_t^S = \frac{\partial V_t^\pi}{\partial S_t} - \lambda_t^O \cdot \frac{\partial O_t^P(d_O, K_t)}{\partial S_t},$$

$$\lambda_t^Z = \frac{\frac{\partial V_t^\pi}{\partial r_t}}{\frac{\partial Z_t(d_Z)}{\partial r_t}} - \lambda_t^O \cdot \frac{\frac{\partial O_t^P(d_O, K_t)}{\partial r_t}}{\frac{\partial Z_t(d_Z)}{\partial r_t}},$$

$$\lambda_t^C = \Psi_t - \left(\lambda_t^S \cdot S_t + \lambda_t^Z \cdot Z_t(d_Z) + \lambda_t^O \cdot O_t^P(d_Z, K_t)\right).$$

While it is used for controlling (implied) volatility risk, the option also gives exposure to the other two considered risk factors, the fund price S_t and the short rate r_t. Therefore, with the amount of options in the hedge portfolio, λ_t^O, being calculated such that the current sensitivity of V_t^π to volatility is matched, λ_t^S and λ_t^Z have to be calculated considering the exposure already attained via the options in order for the hedge portfolio to match the current sensitivity of V_t^π to S_t and to r_t, respectively.

The derivatives of V_t^π are approximated via the "bump and revalue" approach (cf. for instance Cathcart et al., 2015), where we use the central finite difference with regard to the fund price S_t, and the forward and backward finite difference for V_t and r_t, respectively. The same approach is used for the derivatives of the option value.

3 Numerical results

In order to understand the risk profile of a pool of GLWB policies, it helps to analyze the inherent guarantees with regard to their sensitivity to several influencing factors, such as valuation assumptions, market movements, policyholder behavior and demographic risks like longevity. In our analyses, we focus on the purely financial risk drivers equity returns, equity (implied) volatility and interest rates. First, we present the parameters and assumptions used in the subsequent analyses in Section 3.1. In Section 3.2, we present the results of an analysis of the "Greeks" of V_t^π, i.e. the sensitivity of the value of liabilities to the considered financial risk drivers. In Section 3.3, we analyze the risk profile of the pool of policies by means of a simulation study and compute indicators for corresponding capital requirements with regard to the inherent market risk. How this risk profile and, thereby, capital requirements change with variations of interest and volatility levels is analyzed in Section 3.4.

3.1 Parameters and assumptions

For the remainder of this section, we use the following parameters and assumptions, unless stated otherwise.

Market parameters

Table 1 gives the values of the parameters used for the market model – for valuation purposes under the risk-neutral measure Q as well as the parameters used for the real-world projection (under the measure P) of our risk analysis. The parameters under Q are those used by Bacinello et al., 2011, which we also use for the parameters of the real-world projection – this implies the assumption of a market that is risk-neutral with regard to interest-rate risk as well as equity volatility risk. The parameter μ for the equity process under P is chosen similar to Kling et al., 2014.

Parameter	Value
$r_0, \theta_r^P, \theta_r^Q$	0.03
κ_r^P, κ_r^Q	0.60
σ_r^P, σ_r^Q	0.03
$V_0, \theta_V^P, \theta_V^Q$	$(0.2)^2$
κ_V^P, κ_V^Q	1.50
σ_V^P, σ_V^Q	0.40
$\rho_{S,V}$	-0.70
μ	0.03

Table 1: Values of the market and simulation parameters used in the base-case simulation.

Contract parameters

The assumptions regarding the parameters of the variable annuity contract are stated in Table 2. We use annual calculation dates in our analysis, i.e. the set $\mathcal{T}$ represents anniversaries of the contract's inception date.

Parameter	Value
η^{mc}	1.50%
η^{g}	1.00%
ω	3.25%

Table 2: Values of the product parameters used in the simulation.

Compare the results shown in Figure 3 in the following section for the relationship between the value of future guarantee charges (influenced by η^g) and the value of future guarantee payments (influenced by ω). As we do not consider surrender in our analysis, the value of η^s is not relevant and therefore not given here. Also, neither the initial investment amount of a single contract, F_0, nor the number of active contracts at $t = 0$, π_0^A, is relevant, as we state all results as percentages of the pool's total initial investment amount ($\pi_0^A \cdot F_0$).

Policyholder behavior and mortality parameters

As stated in the beginning of Section 2, we assume that no lapses occur in the pool of policies – neither for the purpose of valuation of the liabilities nor for the projection of the pool of policies. That is, we set $s_i = s\,(t_{i-1}, t_i) = 0$ for all $i = 1, \dots, N$.

We also assume the policyholder to withdraw exactly the guaranteed withdrawal amount at each withdrawal date, i.e. we assume there is neither partial surrender nor any kind of deferral or similar delays in withdrawals (withdrawals begin immediately at the first withdrawal date t_1, i.e. one year after inception).

Regarding mortality, we use the best-estimate probabilities for both valuation and the projection of the pool of policies. The persons insured within the pool of policies are assumed to be male and aged 65 at the contract's inception date. We use the best-estimate probabilities for annuities given in the DAV 2004R table published by the German Actuarial Association (DAV). The mortality table has a limiting age (i.e. the age after which survival is assumed to be impossible) of 121, which leads to a maximum projection horizon of 57 years $(122 - 65)$, which also gives the value of N, the last index of the calculation dates.

Hedging parameters

As to the modeling of the hedge instruments, we use $d_O = \frac{3}{12}$, i.e. 3 months for the option's term to maturity, and $d_Z = 10$, i.e. 10 years for the bond's duration. As stated in Section 2.4, the options used in the hedging program have a strike set at the spot price of the underlying fund at the time of buying them, i.e. they are bought "at the money".

In our simulation study, we consider rebalancing of the hedge portfolio (i.e. a hedge frequency) on a monthly and a weekly basis, as well as no hedging at all. The latter gives an indication for the capital requirements if the hedging program is not considered at all.

3.2 Analysis of the "Greeks"

In this first analysis of the risk profile of a pool of variable annuities with GLWB riders, we consider an exemplary pool of homogeneous policies and analyze the Greeks of the pool's value of liabilities at $t = 0$, i.e. V_0^π. We also analyze how these Greeks at inception change if an instantaneous shock is applied to the variable annuity's account value, i.e. if the account value changes a theoretical second after inception. This helps to get a better understanding of the pool's risk profile, as it illustrates how the sensitivity of the pool's value of liabilities changes with market movements. This sensitivity to (market) risk factors directly affects risk measures and, therefore, risk-based capital requirements.

In order to illustrate the impact of the ratchet mechanism that is included in the modeled GLWB rider, we also show results for a stylized GLWB rider without a ratchet mechanism, i.e. a GLWB rider whose guaranteed withdrawal amount is constant and cannot increase after inception. For this GLWB rider without ratchet mechanism, we use the same parameters as given in Table 2, i.e., with the exception of the ratchet mechanism, both products are identical. Note also that this means that, in comparison to the product with ratchet, the product without ratchet is "overpriced".

For all computations in this section, we used 100,000 Monte-Carlo paths under the risk-neutral measure Q. In Figure 1, we show the expected trigger time of the guarantee (i.e. the contract year in which the account value hits zero) as a function of the account value immediately after inception. In other words, we analyze the impact an instantaneous shock to the variable annuity's underlying fund has on the point in time when the insurer has to start guaranteed payments. Note that the analyzed expected trigger time is an expected value under the risk-neutral measure Q, which is why the values are not to be taken as a best estimate, but rather as an explanatory aid to understand the characteristics of the value of liabilities and its corresponding risk profile.

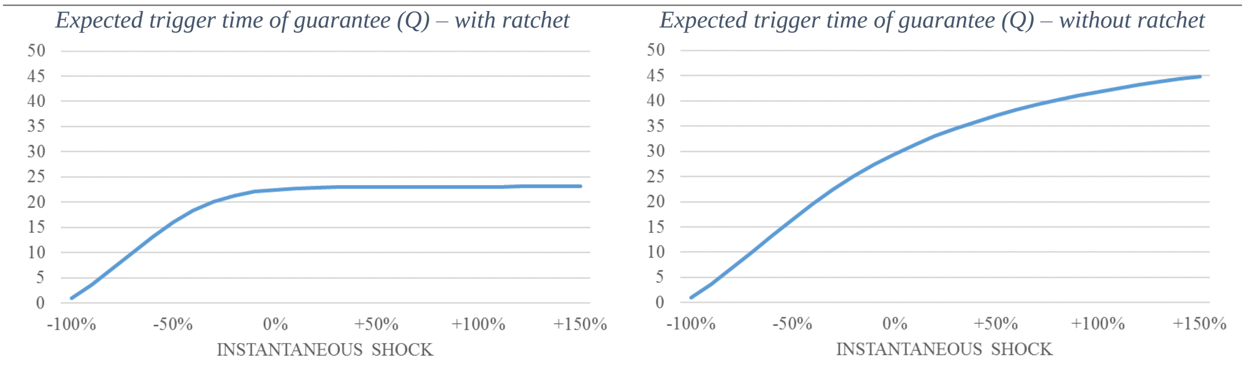

Figure 1: Expected trigger time of the guarantee (time of ruin) in years after inception, analyzed in the context of the valuation under Q for different levels of the instantaneous shock applied to the fund value. With a ratchet mechanism as in the normal design of the GLWB (left-hand side) and, as variation, without ratchet (right-hand side).

In the calculation of the expected trigger time, the trigger time of the guarantee is taken into account as $57\ (122 - 65)$ if the guarantee does not trigger before the insured person reaches the limiting age of the mortality table. For the variable annuity without ratchet (right-hand side in Figure 1), the higher the account value the later the guarantee triggers, i.e. the later the account value hits zero. For the design with ratchet (left-hand side in Figure 1), a higher account value also leads to higher guaranteed withdrawal benefits (due to the ratchet mechanism), which counteracts the prolonging effect and leads to a “top out” of the expected trigger time at around 23 years.

In Figure 2, we show similar plots of the value of liabilities and its two components, the value of future guarantee payments (to be made by the insurer) and the value of future guarantee charges (to be received by the insurer). We also show plots of the derivatives of these values: the derivative with regard to the underlying fund’s share price (the Delta) multiplied by the share price (i.e. the “Dollar-Delta” or “EUR-Delta” of V_0^{π}), the derivative with regard to the short rate (similar to Rho in the Black-Scholes-Merton model) and the derivative with regard to the instantaneous volatility (similar to Vega in the Black-Scholes-Merton model). All values are stated as a fraction of the initial investment amount, i.e. as a percentage of $\pi_0^A \cdot F_0$.

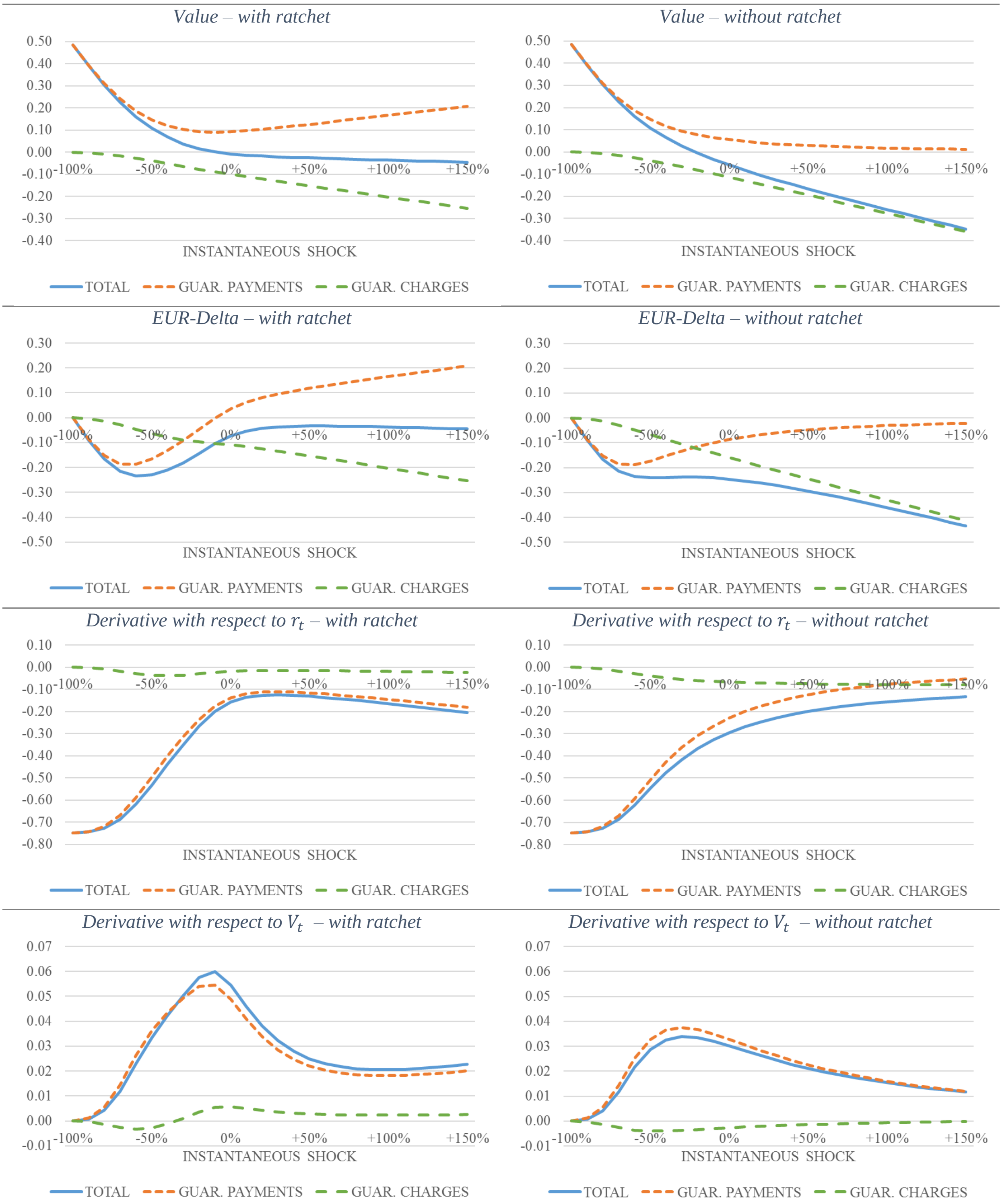

Figure 2: Value and "Greeks" at $t = 0$ as a function of the instantaneous shock applied to the account value. Charts on the left-hand side show the results for a GLWB design with a ratchet mechanism and charts on the right-hand side results for a GLWB design without ratchet (for comparison purposes). Values are stated as a fraction of the initial investment amount.

Note that Figure 2 shows values as liabilities, i.e. future guarantee charges have a negative value, as they represent future earnings of the insurer. Intuitively, if the account loses value, the value of liabilities increases. When the account value increases, however, it is not intuitively clear what happens, as now the GLWB's ratchet comes into play. Without a ratchet mechanism, the value of liabilities would strictly decrease, as future guarantee payments become less likely and future guarantee charges become more valuable. With a ratchet in place, however, positive fund returns can potentially lead to increases in the value of future guarantee payments, due to an increase in the guaranteed withdrawal amount. Which of these two effects dominates the value of liabilities depends on several factors, including the pricing and the design of the GLWB rider, as well as other factors relevant for the valuation, e.g. interest rates and volatility. In the considered case, the value of future guarantee charges increases faster than the value of future guarantee payments. As a result, the total value of liabilities slightly decreases if the fund value increases.

For the GLWB rider without ratchet, the EUR-Delta of the guarantee payments is similar to the EUR-Delta of a simple put option. For the GLWB rider with a ratchet, however, the EUR-Delta of the guarantee payments becomes positive if the account value increases, reflecting the increase in value caused by the ratchet mechanism. In comparison, the ratchet also causes the value of future guarantee charges to increase (from the insurer's perspective) less with higher account values, because over time, higher account values are also reduced by higher future withdrawal amounts. Therefore, the EUR-Delta of future guarantee charges is less pronounced for the rider with ratchet. Also, for the product with ratchet, the EUR-Delta of future guarantee charges more or less counter-balances the EUR-Delta of future guarantee payments for higher account values, resulting in a total EUR-Delta of near zero in good scenarios (as opposed to the product without ratchet, where the total EUR-Delta in good scenarios is dominated by the sensitivity of the value of future guarantee charges). The increase in value caused by the ratchet mechanism is also visible in the sensitivity to the short rate and the instantaneous volatility: in both cases, there is a slight increase in sensitivity at the positive end of the considered fund shocks.

For both riders, the sensitivity to the short rate sharply increases with increasing "moneyness" of the guarantee, i.e. with declining account values. This is also similar to the pattern of the Rho of a simple put option. An adverse market movement therefore causes a simultaneous increase in the exposure to interest rate risk – and therefore likely an increase in corresponding risk-based capital requirements. However, in the base position with an unchanged account value, the sensitivity to the short rate seems rather low. This may also be due to the choice of parameters for the interest-rate model, where lower values for the speed of mean-reversion or higher interest-rate volatility could lead to significantly higher sensitivities to the short rate.

The sensitivity of the value of liabilities to the instantaneous volatility shows the typical pattern of a bump around the "strike" of the guarantee. The sensitivity to the instantaneous volatility is similar for both considered riders if the account loses more than 50% (i.e. for shocks between -100% and -50%) but changes as soon as the likelihood for the ratchet having an effect increases, with the design with ratchet showing a much more pronounced sensitivity to volatility. This is in line with the findings in Kling et al., 2011, in which the impact of stochastic volatility on different product designs is analyzed. Also, for the rider with ratchet, the value of future guarantee charges shows a change of sign, causing an overall even higher sensitivity to volatility of the value of liabilities for moderate to positive shocks of the account value.

Figure 3 shows a similar analysis, but now, instead of the account value, the GLWB's guaranteed withdrawal rate ω is changed.

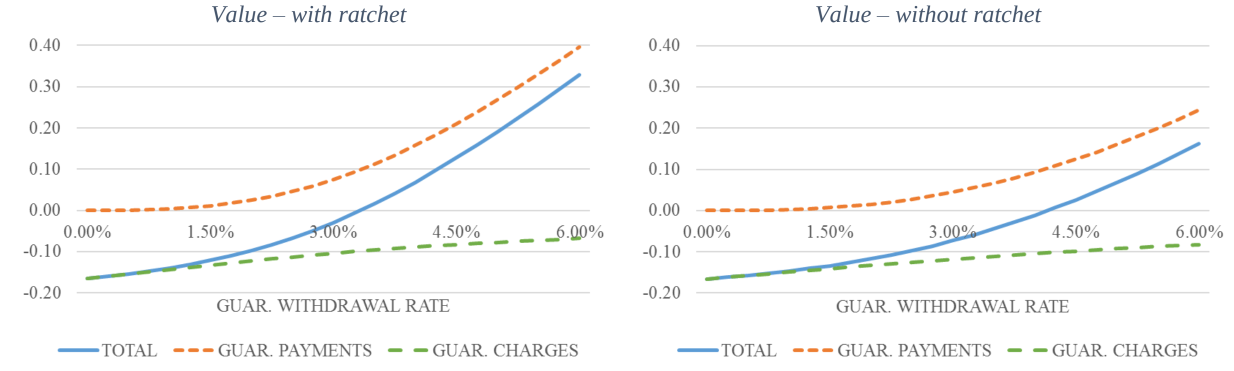

Figure 3: Value (as fraction of the initial investment amount) at $t = 0$ for different levels of the guaranteed withdrawal rate ω.With a ratchet mechanism as in the normal design of the GLWB (left-hand side) and as a variation without ratchet (right-hand side).

Finding the level of ω for which the total value is (close to) zero is typically part of the pricing process. For instance, at a guaranteed withdrawal rate of 3.25%, the total value of liabilities (from the insurer's perspective) amounts to -0.8% of the initial investment for the GLWB with ratchet, where the value of future guarantee payments of 9.2% is (more than) compensated by the value of future guarantee charges of -10.0%. This guaranteed withdrawal rate, $\omega = 3.25\%$, is the pricing assumption used in the following analyses of the risk profile (cf. Table 2).

Of course, the value of the guaranteed payments is zero if ω is zero. For positive values of ω, the impact of the ratchet is clearly visible: the value of future guarantee payments increases much faster if the GLWB has a ratchet mechanism. This results from the withdrawal rate affecting two components at once: the minimum guaranteed withdrawal amount at inception and the amplitude of potential future increases of the guaranteed withdrawal amount due to the ratchet. Note that with the considered ratchet design (cf. Section 2.1), the fund performance only has to compensate for charges, not for withdrawals (within the guaranteed limit), and, thus, the impact of higher withdrawal rates on the potential for future ratchets is reduced.

3.3 Analysis of Risk Profile and Capital Requirements

In this section, we analyze the distribution of the insurer's one-year P&L with regard to the modeled pool of GLWB policies. Using this distribution, we calculate risk metrics and, thereby, indicators for risk-based capital requirements, as, for instance, stipulated under Solvency II (99.5%-Value-at-Risk on a one-year basis) or the Swiss Solvency Test (99%-Tail-Value-at-Risk, also on a one-year basis). Using the results for the 99.5%-Value-at-Risk, our approach is comparable to a Solvency II internal model type approach (cf. e.g. Central Bank of Ireland, 2010).

We assume different levels of consideration of the insurer's hedging program, ranging from no credit at all (i.e. no allowance of the risk-mitigating effect of future hedging), over hedging with monthly rebalancing to weekly reallocations of the hedge portfolio. In reality, the "Future Trading Offset", i.e. the difference in capital requirements calculated without and with (full) consideration of the risk-mitigating effect of future hedging, is likely not allowed to be fully applied when calculating the capital requirements. Instead, it will likely only be partially considered, taking into account (amongst others) the effectiveness of the hedging program (considering e.g. basis risk) and how close to reality the modeling of the hedging program is (cf. e.g. Central Bank of Ireland, 2010).

We used 10,000 runs (paths) under the real-world measure P for each simulation in this section. Within each of these 10,000 real-world paths, we used 10,000 risk-neutral paths under the measure Q in order to re-evaluate the liabilities at the end of the one-year projection as well as 1,000 risk-neutral paths for the calculation of the Greeks at each rebalancing date of the modeled hedging program. We only consider the product design with a ratchet mechanism in this analysis.

Distribution of the present value of liabilities

First, we have a look at the distribution of the (discounted) one-year change of the value of liabilities, stated as a fraction of the initial investment amount, i.e. the pool's assets under management at the beginning of the year. In formal terms, we look at the distribution of

$$\frac{\frac{C_0}{C_1}V_1^{\pi} - V_0^{\pi}}{\pi_0^A \cdot F_0}.$$

Figure 4 shows scatter plots in which this one-year change is plotted against the fund's one-year return, $\frac{S_1}{S_0} - 1$, the short rate at the end of the year (r_1), and the instantaneous volatility at the end of the year ($\sqrt{V_1}$), respectively.

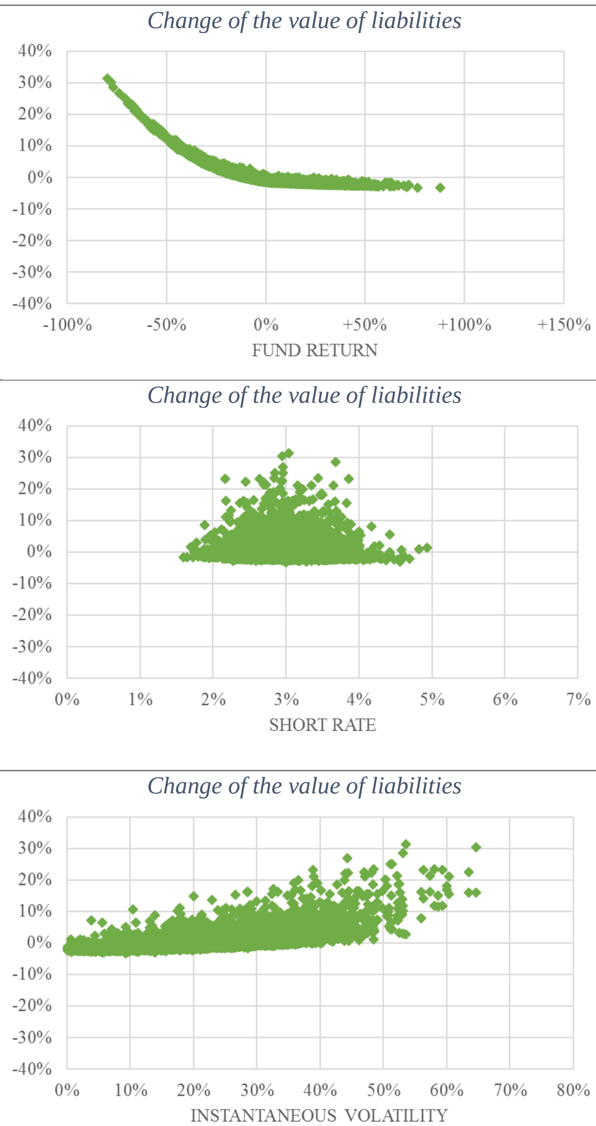

Figure 4: Scatter plots of the discounted absolute change (as percentage of the investment amount at the beginning of the year) of the present value of liabilities (value of future guarantee payments minus future guarantee charges) as a function of the one-year fund return, the short rate at the end of the year and the instantaneous volatility, also at the end of the year. 10,000 paths of the base-case simulation.

Of course, the value of liabilities is strongly affected by the underlying fund's one-year return, with a maximum increase of around 30% of the initial AuM in the worst path. The value is strictly decreasing with respect to positive fund returns and shows the same pattern as in Figure 2.

In comparison, the short rate at the end of the year, r_1, seems to have a negligible effect, as there is no clear influence of r_1 on the level of changes – the higher levels of changes around the mean-reversion level of 3% can be explained by the majority of the runs lying in that area. This is in line with the results in Section 3.2, where, at inception, the sensitivity of the value of liabilities to the short rate is rather low – as long as the fund value hasn't decreased significantly. Also, note that, while the negative correlation between equity returns and instantaneous volatility has a systematic cumulative effect on risk, the Wiener process of the short rate is independent of the other two and, thus, there is no similar cumulative effect with equity returns and the short rate.

Ceteris paribus, the higher the volatility, the higher the value of liabilities – the derivative with respect to the instantaneous volatility is non-negative, irrespective of the current account value (see Figure 2). This dependence is also visible in the scatter plot against the instantaneous volatility at the end of the year, i.e. against $\sqrt{V_1}$. Another contributing factor is the modeled negative correlation between the instantaneous volatility and equity returns, which increases both, the dependence between the value of liabilities and the instantaneous volatility at the end of the year as well as the dependence between the value of liabilities and the fund's one-year return.

Next, we analyze how well the modeled hedging programs are capable of replicating the observed changes in the value of liabilities.

Hedge performance

In order to assess the hedge performance, we look at the (discounted) P&L at the end of the projected year, stated as a percentage of the pool's assets under management at the beginning of the year. We denote this value by $\Pi_{0,1}$ and define it as follows:

$$\Pi_{0,1} := \frac{\frac{C_0}{C_1}(\Psi_1 - V_1^\pi) - (\Psi_0 - V_0^\pi)}{\pi_0^A \cdot F_0}.$$

This means that $\Pi_{0,1}$ is the (discounted) net P&L at the end of the projection year, i.e. the (discounted) change of the difference in value of the hedge portfolio and the value of liabilities (as a percentage of the initial AuM).

We also look at the absolute discounted change of the hedge portfolio separately, i.e. we consider the value

$$\frac{\frac{C_0}{C_1}\Psi_1 - \Psi_0}{\pi_0^A \cdot F_0}.$$

We assume two different rebalancing frequencies of the hedging program: monthly and weekly. We also consider the resulting P&L if no hedging program is in place or the existing hedging program is not recognized for the purpose of calculating the risk-based capital requirements.

Weekly rebalancing

Figure 5 shows scatter plots of the one-year change in the value of the hedge portfolio with assumed weekly rebalancing of the hedge portfolio (left-hand side), as well as the scatter plots for the resulting P&L $\Pi_{0,1}$ (right-hand side).

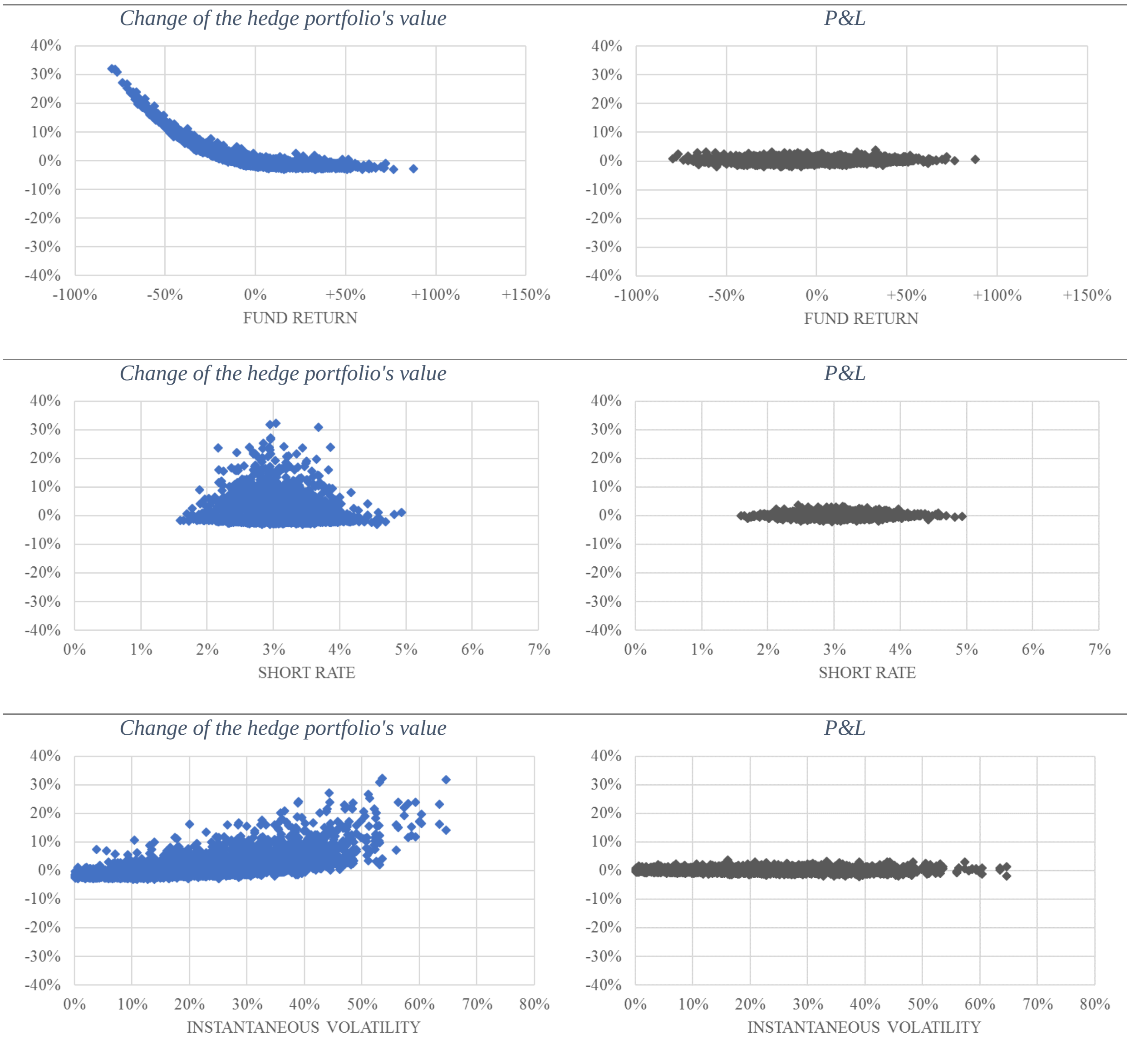

Figure 5: Scatter plots of the discounted absolute change (as percentage of the investment amount at the beginning of the year) of the hedge portfolio's value over the one-year projection (left-hand side) and resulting discounted net P&L (right-hand side), both as a function of the one-year fund return, the short rate at the end of the year and the instantaneous volatility, also at the end of the year. 10,000 paths of the base-case simulation with weekly rebalancing of the hedge portfolio.

The modeled hedging program seems to effectively offset the change of the value of liabilities, resulting in no visible dependency between the P&L and the fund's return. The resulting P&L also does not seem to be dependent on the short rate or the instantaneous volatility at the end of the year.

Monthly rebalancing

Figure 6 shows scatter plots of the one-year change in the value of the hedge portfolio with assumed monthly rebalancing (left-hand side), as well as the resulting P&L $\Pi_{0,1}$ (right-hand side).

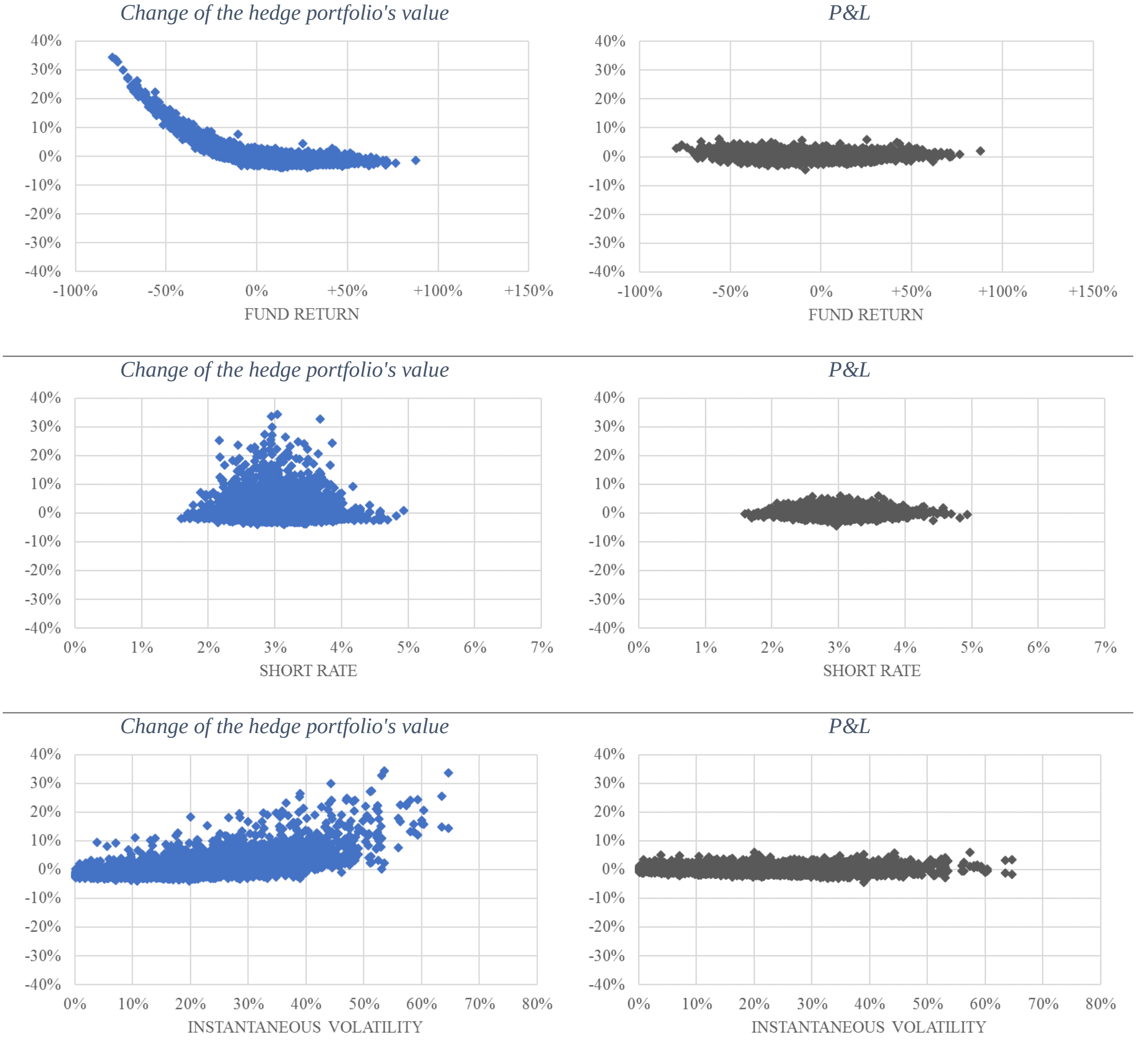

Figure 6: Scatter plots of the discounted absolute change (as percentage of the investment amount at the beginning of the year) of the hedge portfolio's value over the one-year projection (left-hand side) and resulting discounted net P&L (right-hand side), both as a function of the one-year fund return, the short rate at the end of the year and the instantaneous volatility, also at the end of the year. 10,000 paths of the base-case simulation with monthly rebalancing of the hedge portfolio.

In comparison to the results for weekly rebalancing, the replication is less accurate, resulting in a broader distribution of the P&L. The overall impact of the considered risk factors still seems to be rather limited, but the higher variance of the P&L is noticeable when comparing the resulting risk measures (see next section).

No Hedge

We also analyze the resulting P&L if no hedging program is in place or an existing hedging program is not considered in the calculation of the capital requirements. In this case, the change of the hedge portfolio's (discounted) value is zero in all paths and the net P&L equals the (discounted) negative of the change in the value of liabilities (cf. Figure 4).

Comparison of the P&L distribution and risk measures

In Figure 7, we have a look at the empirical (cumulative) density functions of $\Pi_{0,1}$ for the three different modeled hedging programs.

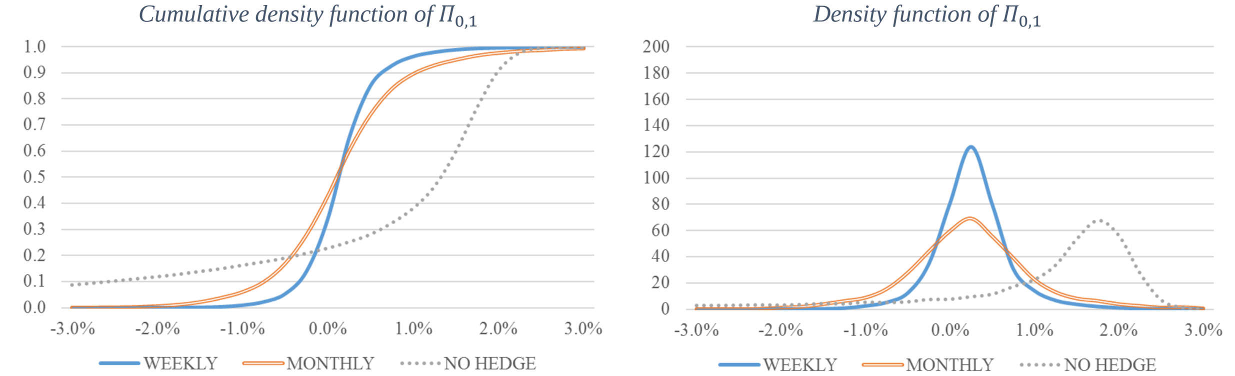

Figure 7: The (cumulative) density function of the discounted net P&L $\Pi_{0,1}$ for all three considered variations of the modeled hedging program (weekly rebalancing, monthly rebalancing and no hedging at all). Based on 10,000 paths of the base-case simulation.

For the hedging programs with weekly and monthly rebalancing, most of the distribution's mass is located between -2% and +2%, with the monthly rebalancing showing a higher deviation than the weekly rebalancing. If no hedging program is considered, around 10% of the paths result in a P&L of less than -3%, whereas most of the paths seem to result in a P&L that is located around +2%.

These patterns can also be found in the results shown in Table 3, in which (risk) measures of $\Pi_{0,1}$, including the value-at-risk (VaR) and the conditional-value-at-risk (CVaR), are shown.

Characteristics of $\Pi_{0,1}$	Weekly Hedge	Monthly Hedge	No Hedge
mean	0.1%	0.1%	0.3%
median	0.1%	0.1%	1.3%
standard deviation	0.4%	0.8%	2.8%
$\mathbf{VaR_{99.5\%}}$	1.2%	2.1%	16.0%
$\mathbf{CVaR_{99.5\%}}$	1.4%	2.5%	20.2%
$\mathbf{VaR_{95\%}}$	0.5%	1.1%	5.0%
$\mathbf{CVaR_{95\%}}$	0.8%	1.6%	9.3%
$\mathbf{VaR_{90\%}}$	0.3%	0.7%	2.6%
$\mathbf{CVaR_{90\%}}$	0.6%	1.2%	6.5%
$\mathbf{VaR_{99.5\%}} + V_0^{\pi}$	0.4%	1.3%	15.2%

Table 3: Risk measures of the discounted net P&L, $\Pi_{0,1}$, for all three considered variations of the modeled hedging program (weekly rebalancing, monthly rebalancing and no hedging at all). Based on 10,000 paths of the base-case simulation.

Depending on the chosen risk measure, the difference between monthly and weekly rebalancing makes up to a factor equal or greater than 2 (for a confidence level of 90% or 95%, respectively) and around 1.75 for a confidence level of 99.5%. If no hedging program is considered, the risk measures are, of course, considerably increased. If we use the results for no hedging as a benchmark to compute a measure of hedge effectiveness, we obtain – using the value-at-risk at a confidence level of 99.5% – a hedge effectiveness of 93% for the weekly hedge ($\frac{16.0\% - 1.2\%}{16.0\%}$, cf. Morrison & Tadrowski, 2015) and 87% ($\frac{16.0\% - 2.1\%}{16.0\%}$) for the monthly hedge.

With regard to risk-based capital requirements, the results mean that – depending on the allowance of the risk-mitigating effect of future hedging – the insurer would be required to hold additional 1.2% to 16.0% of the initial investment amount as capital for the considered market risks (using the value-at-risk at a confidence level of 99.5%). If the rebalancing of the hedging program is only considered on a monthly basis (for instance due to delays in trading or an otherwise slow reaction time), the capital requirements are almost doubled in comparison to weekly rebalancing. Note that these measures only consider the three purely financial risk drivers equity returns, interest rates and equity implied volatility – the results, thus, only represent a part of the total capital requirements for all relevant risks.

We also give the sum of the value-at-risk at a level of 99.5% and the value of liabilities (V_0^{π}) in the last row of Table 3, in order to provide a better understanding of the total impact on the balance sheet of the insurer – this will be relevant in the next section, where – in order to gain a better understanding how the results are influenced by the parameter assumptions made in Section 3.1 – we conduct similar analyses for different assumptions regarding the interest-rate level as well as the level of equity volatility.

3.4 Impact of interest rate and volatility levels on the risk profile

After inception of the variable annuity, the insurer usually aims to offset the changes of the value of liabilities via a hedging program. However, even if a well-functioning hedging program – that is able to effectively offset the changes of the value of liabilities – is in place, the insurer still faces the risk of having to increase its capital resources due to changes in risk-based capital requirements, driven by changes in market parameters. Therefore, we analyze how the indicators for risk-based capital requirements examined in the previous section change with different levels of interest rates and volatility.

We consider four variations of the market environment: higher and lower interest, as well as higher and lower equity volatility. In all four considered variations, we simultaneously change r_0, θ_r^P and θ_r^Q (cf. Table 1) for different interest-rate levels and do the same with V_0, θ_V^P and θ_V^Q in the variations with different levels of equity volatility. See Table 4 for the used parameter values.

Variation	**Parameter**	**New value**
Lower interest	$r_0, \theta_r^P, \theta_r^Q$	0.015
Higher interest	$r_0, \theta_r^P, \theta_r^Q$	0.045
Lower volatility	$V_0, \theta_V^P, \theta_V^Q$	$(0.1)^2$
Higher volatility	$V_0, \theta_V^P, \theta_V^Q$	$(0.3)^2$

Table 4: Values of the market and simulation parameters used in the variations. All other parameters are as in the base case.

With the changed parameter values for interest rates and equity volatility, respectively, we conduct similar analyses as in the previous section. We also use the same pricing as previously, i.e. the same guarantee charge and the same guaranteed withdrawal rate as stated in Table 2, meaning that the value of the GLWB rider at inception changes. This represents either a situation where the market environment changes a theoretical second after inception of the contract (with the change in the value of liabilities potentially being hedged) or a situation where the pricing for the offered GLWB rider is only updated after a certain time period and hence does not (immediately) reflect changes in the market environment. In the latter situation, from the insurer’s perspective, the value of liabilities of newly sold contracts with otherwise identical parameters fluctuates. Table 5 gives an overview over the value of liabilities at inception, as well as its components, the value of future guarantee payments and the value of future guarantee charges.

Variation	**Value guar. payments**	**Value guar. charges**	**Total value (V_0^π)**
Base case	9.2%	-10.0%	-0.8%
Lower interest	14.6%	-9.7%	4.9%
Higher interest	5.7%	-10.2%	-4.5%
Lower volatility	3.2%	-10.4%	-7.2%
Higher volatility	17.2%	-9.6%	7.6%

Table 5: Resulting values of liabilities at inception and its components for the considered parameter variations. Values are given as fraction of the invested amount at inception.

For instance, in the variation with a lower interest-rate level, the total value of liabilities is 4.9% with the value of future guarantee payments of 14.6% not being compensated by the value of future guarantee charges of -9.7%. While the value of future guarantee charges is rather stable, the value of future guarantee payments shows noticeable differences between the considered variations, ranging between 3.2% and 17.2% of the pool's AuM at the beginning of the year.

We do not show all scatter plots shown in Section 3.3, but instead only show the scatter plots for the change of the value of liabilities (Figure 8, Figure 9 and Figure 10), the comparison of the (cumulative) density functions (Figure 11 and Figure 12), as well as the resulting risk measures (Table 6 and Table 7) for each variation.

Distribution of the present value of liabilities – variations

Figure 8 shows scatter plots of the one-year change of the value of liabilities plotted against the underlying fund's one-year return in all four considered parameter variations (cf. Figure 4 for corresponding plots of the base-case parametrization).

Figure 8: Scatter plots of the discounted absolute change (as percentage of the invested amount at the beginning of the year) of the present value of liabilities (value of future guarantee payments minus future guarantee charges) as a function of the one-year fund return. Based on 10,000 simulation paths for each parameter variation.

The effect of the volatility's parameterization on the distribution of the fund's return is clearly observable, with the width of the fund returns significantly widening with higher volatility. With the lower volatility level, the fund returns only reach around +50%, while they reach up to +150% in the simulation with a higher volatility level.

In a lower interest-rate environment, the increase in future guarantee payments (due to the ratchet) is more valuable than the increase in future guarantee charges, resulting in an increase in the value of liabilities with positive fund returns. The opposite can be noticed in the variation with higher interest rates: here the ratchet has a lower value and, thus, the value of liabilities decreases with increasing fund returns. Similar effects can be observed with the volatility variations: While the change in value for negative fund returns is similar for both volatility levels, the value increases with positive fund returns for the higher volatility level and decreases for the lower volatility level.

Figure 9: Scatter plots of the discounted absolute change (as percentage of the invested amount at the beginning of the year) of the present value of liabilities (value of the future guarantee payments minus future guarantee charges) as a function of the short rate at the end of the year. Based on 10,000 simulation paths for each parameter variation.

The scatter plots against the short rate r_1 (shown in Figure 9) show a much broader lateral scatter in the variation with higher interest rates, which illustrates the CIR model's volatility structure, where (ceteris paribus) higher levels of interest are accompanied by higher levels of interest-rate volatility. Still, the influence of r_1 on the overall distribution of the value of liabilities seems to be of secondary nature.

Similarly, Figure 10 shows scatter plots of the one-year change of the value of liabilities with respect to the instantaneous volatility at the end of the projection year, $\sqrt{V_1}$.

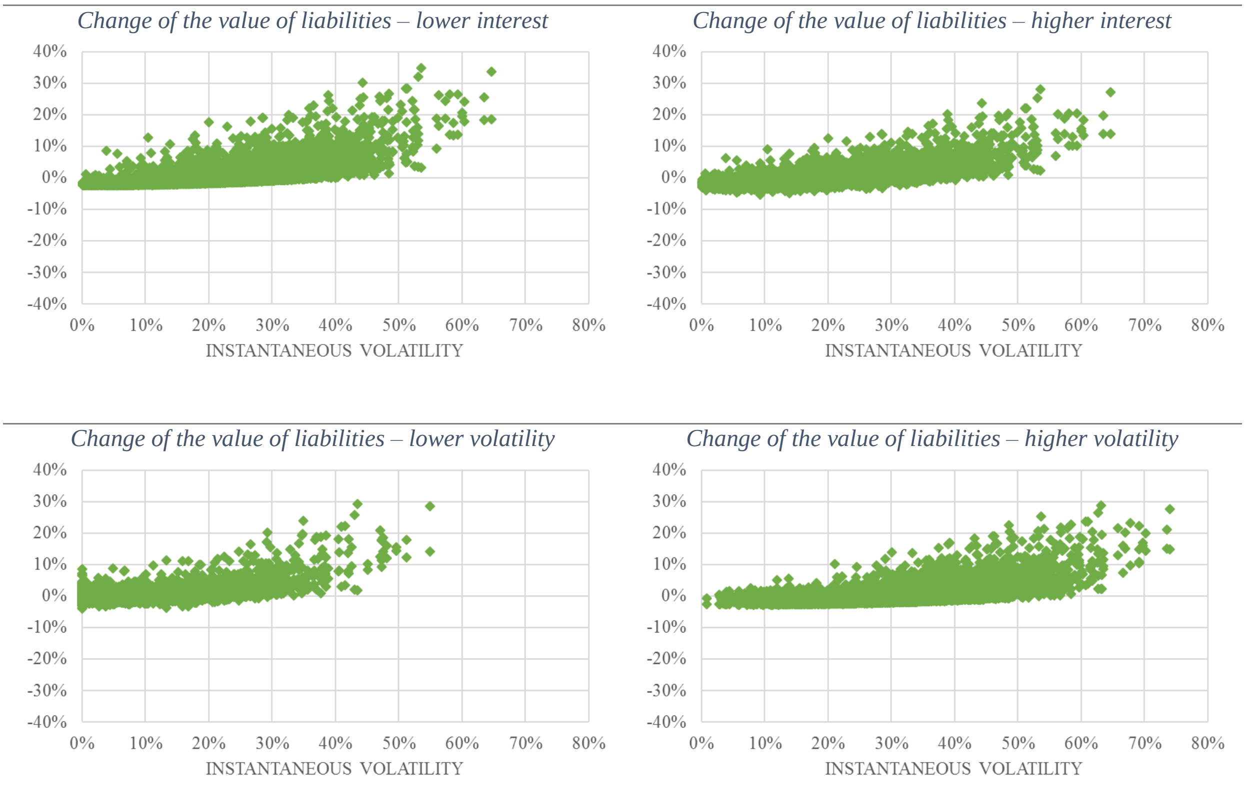

Figure 10: Scatter plots of the discounted absolute change (as percentage of the invested amount at the beginning of the year) of the present value of liabilities (value of the future guarantee payments minus future guarantee charges) as a function of the instantaneous volatility at the end of the year. Based on 10,000 simulation paths for each parameter variation.

For the two interest-rate variations, the scatter plots against the instantaneous volatility seem to be similar to the base case, with the importance of volatility in the "good scenarios" (reduction in the value of liabilities, i.e. the lower edge in the scatter plots), being higher if interest rates are low.

Similar to the higher volatility of interest-rates in the variation with a higher level of interest, an increase of the level of volatility also causes an increase of the observed volatility of volatility (i.e. a wider lateral scatter of $\sqrt{V_1}$), potentially making hedging much harder. However, this effect does not seem to be very pronounced in the variation with a higher volatility level, with the highest observed values of $\sqrt{V_1}$ only shifting by around 10 percentage points in comparison to the base case (which is the same as the increase in the volatility level).

It is noticeable that, in the variation with a lower volatility level, there seem to be quite a few paths where the instantaneous volatility reaches (or gets very close) to zero. This is to be expected, as the parameters in this variation clearly do not fulfill the condition under which V_t cannot reach zero (${\sigma_V^P}^2 > 2\kappa_V^P\theta_V^P$), therefore zero is accessible (cf. Cox et al., 1985).

Comparison of the P&L distribution and risk measures – variations

The (cumulative) density functions given in Figure 11 show a considerably greater deviation if the interest-rate level is lower, accounting for the increase in the value of liabilities and the increase in the guarantee's "moneyness", as the fund's performance now is less likely to be able to compensate for withdrawals and, thus, the guarantee is more likely to trigger. The opposite can be said for the variation with higher interest, where a narrower distribution of the P&L can be observed. Note that $\Pi_{0,1}$ only measures the P&L after inception of the contract (i.e. the deviation) and does not reflect the initial impact on the balance sheet when the value of liabilities V_0^π is added. Hence, a potential considerable loss or profit from V_0^π is not shown in the density functions and, as a consequence, there is no considerable shift to the left or right.

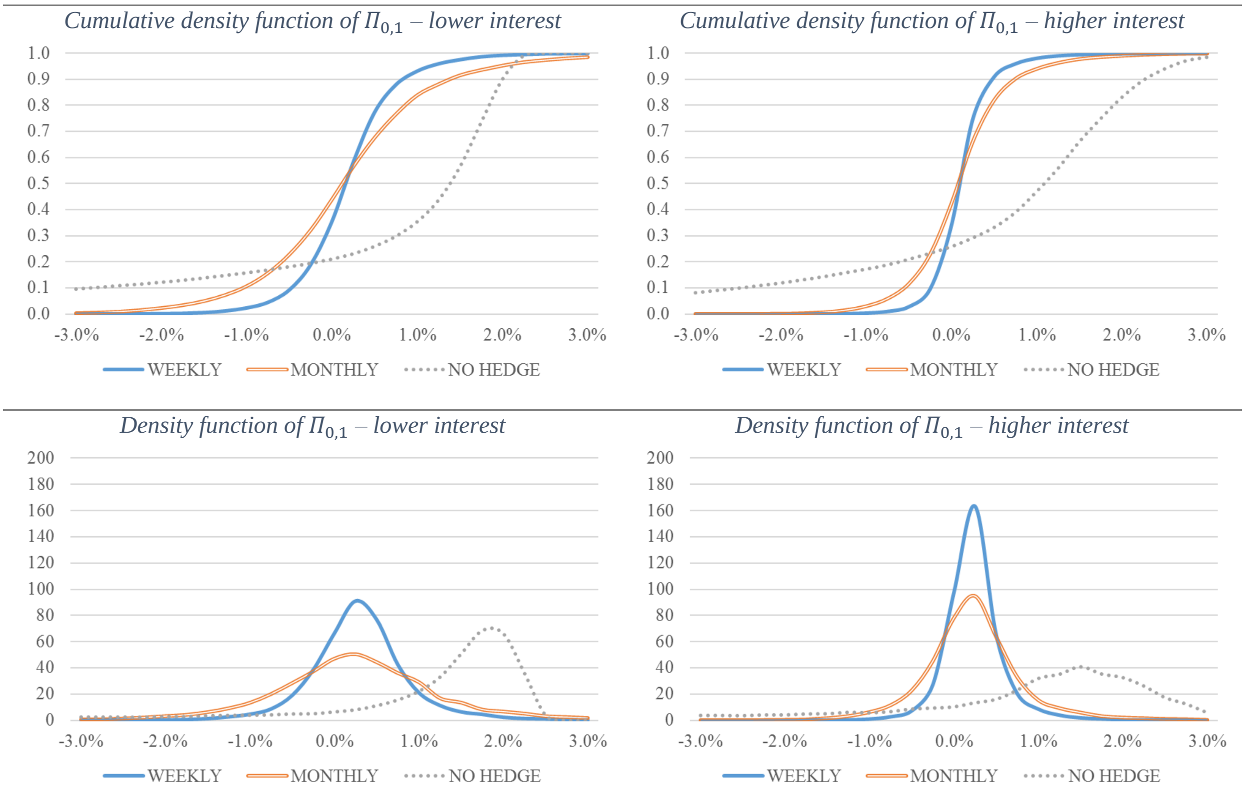

Figure 11: The (cumulative) density function of the discounted net P&L $\Pi_{0,1}$ for all three considered variations of the modeled hedging program (weekly rebalancing, monthly rebalancing and no hedging at all). Based on 10,000 paths of the simulation with a lower (left-hand side) and a higher interest-rate level (right-hand side).

Similar effects can be observed with the equity-volatility variations, with the higher volatility level leading to a significant broadening of the respective density functions (cf. Figure 12). Judging from the density functions in the lower volatility variation, the difference between monthly and weekly hedging seems much less pronounced than in the other considered market environments. However, as can be observed in Table 7, this is only true for the body of the distribution, and not for its tail.

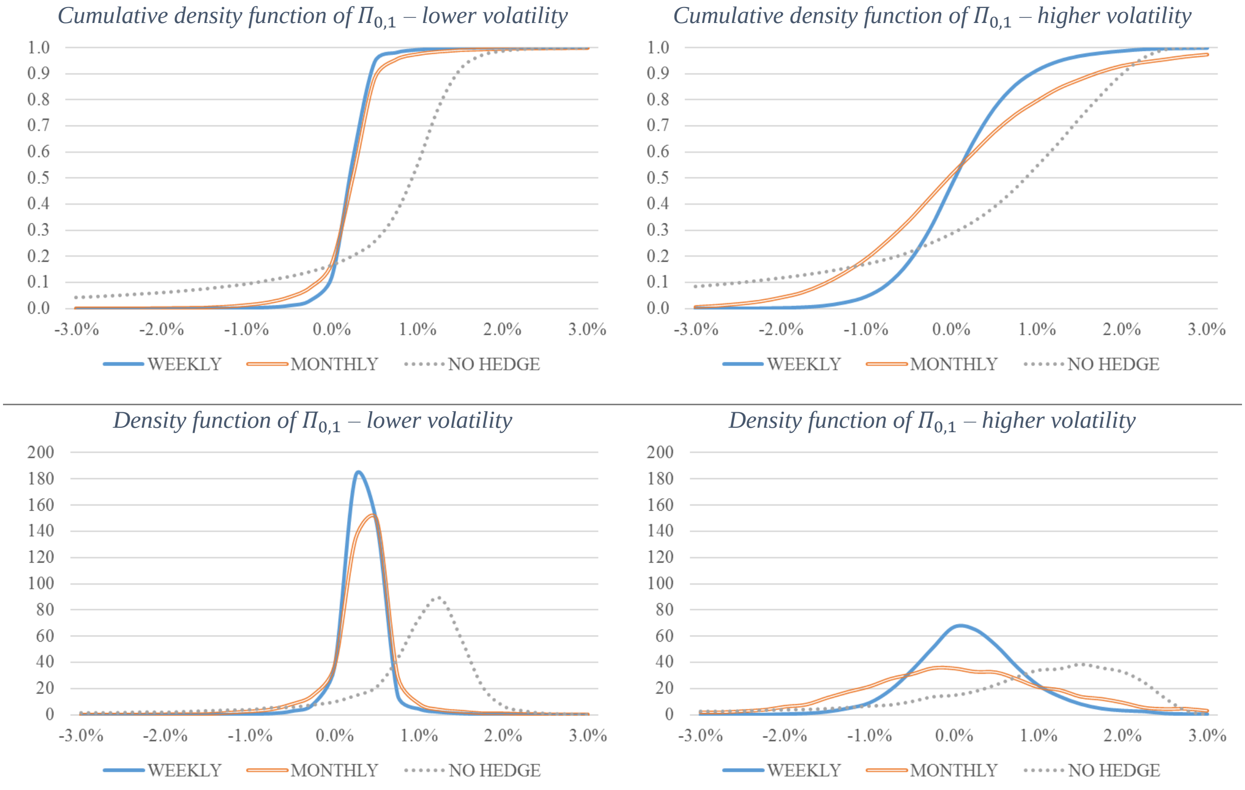

Figure 12: The (cumulative) density function of the discounted net P&L $\Pi_{0,1}$ for all three considered variations of the modeled hedging program (weekly rebalancing, monthly rebalancing and no hedging at all). Based on 10,000 paths of the simulation with a lower (left-hand side) and a higher equity-volatility level (right-hand side).

The risk measures given in Table 6 show that higher interest rates not only reduce the value of liabilities, but also reduce risk measures and, thereby, risk-based capital requirements. In the lower interest-rate environment, however, the greater deviation of the P&L is also noticeable in the resulting risk measures – meaning that, additionally to the value of liabilities increasing from -0.8% to 4.9% (see Table 5), the risk measures increase and, thus, lead to higher capital requirements. From the insurer's perspective, this means a "double hit", increasing both the value of liabilities and the corresponding capital requirements. For instance, the sum of the value-at-risk at a confidence level of 99.5% and the value of liabilities V_0^π increases from 0.4% to 6.4% if weekly hedging is considered and from 15.2% to 23.3% without hedging.

Characteristics of $\Pi_{0,1}$	**Weekly Hedge**			**Monthly Hedge**			**No Hedge**		
Interest level	lower	base	higher	lower	base	higher	lower	base	higher
mean	0.2%	0.1%	0.1%	0.2%	0.1%	0.1%	0.3%	0.3%	0.3%
median	0.2%	0.1%	0.1%	0.1%	0.1%	0.1%	1.4%	1.3%	1.1%
std. deviation	0.6%	0.4%	0.3%	1.1%	0.8%	0.6%	3.2%	2.8%	2.6%
$\mathbf{VaR_{99.5\%}}$	1.5%	1.2%	0.9%	2.7%	2.1%	1.6%	18.4%	16.0%	13.7%
$\mathbf{CVaR_{99.5\%}}$	1.8%	1.4%	1.1%	3.2%	2.5%	1.9%	23.2%	20.2%	17.6%
$\mathbf{VaR_{95\%}}$	0.7%	0.5%	0.4%	1.5%	1.1%	0.8%	5.8%	5.0%	4.5%
$\mathbf{CVaR_{95\%}}$	1.1%	0.8%	0.6%	2.1%	1.6%	1.2%	10.9%	9.3%	8.1%
$\mathbf{VaR_{90\%}}$	0.5%	0.3%	0.2%	1.0%	0.7%	0.5%	2.8%	2.6%	2.5%
$\mathbf{CVaR_{90\%}}$	0.8%	0.6%	0.4%	1.7%	1.2%	0.9%	7.5%	6.5%	5.7%
$\mathbf{VaR_{99.5\%} + V_0^\pi}$	6.4%	0.4%	-3.6%	7.6%	1.3%	-2.9%	23.3%	15.2%	9.2%

Table 6: Risk measures of the discounted net P&L $\Pi_{0,1}$ for all three considered variations of the modeled hedging program (weekly rebalancing, monthly rebalancing and no hedging at all) and all three variations of the interest-rate level. Based on 10,000 paths in each corresponding simulation.

The measured hedge effectiveness, again using the results for no hedging as a benchmark and the value-at-risk at a confidence level of 99.5%, is only slightly affected by the change in interest rates, ranging around 92% (lower interest) to 93% (higher interest) for the weekly hedge and 85% (lower interest) to 88% (higher interest) for the monthly hedge.

Similar to the decrease in interest rates, an increase of the volatility level leads to a significant increase in the deviation of the P&L and, thereby, to an increase of the potential capital requirements derived from the considered risk measures. The value-at-risk at a level of 99.5%, for instance, increases from 1.2% to 1.7% if a hedging program with weekly rebalancing is considered (cf. Table 7). This comes additionally to the increase of the value of liabilities from -0.8% to 7.6% (cf. Table 5).

Characteristics of $\Pi_{0,1}$	**Weekly Hedge**			**Monthly Hedge**			**No Hedge**		
Volatility level	lower	base	higher	lower	base	higher	lower	base	higher
mean	0.2%	0.1%	0.1%	0.2%	0.1%	0.1%	0.4%	0.3%	0.1%
median	0.2%	0.1%	0.0%	0.2%	0.1%	0.0%	0.9%	1.3%	0.9%
std. deviation	0.3%	0.4%	0.7%	0.4%	0.8%	1.3%	1.9%	2.8%	2.8%
$\mathbf{VaR_{99.5\%}}$	0.7%	1.2%	1.7%	1.3%	2.1%	3.1%	11.8%	16.0%	15.3%
$\mathbf{CVaR_{99.5\%}}$	1.1%	1.4%	2.0%	1.7%	2.5%	3.5%	16.7%	20.2%	19.6%
$\mathbf{VaR_{95\%}}$	0.2%	0.5%	1.0%	0.4%	1.1%	1.9%	2.6%	5.0%	5.0%
$\mathbf{CVaR_{95\%}}$	0.4%	0.8%	1.3%	0.8%	1.6%	2.4%	6.2%	9.3%	9.4%
$\mathbf{VaR_{90\%}}$	0.0%	0.3%	0.7%	0.2%	0.7%	1.5%	0.9%	2.6%	2.5%
$\mathbf{CVaR_{90\%}}$	0.2%	0.6%	1.1%	0.6%	1.2%	2.0%	3.9%	6.5%	6.4%
$\mathbf{VaR_{99.5\%}} + \boldsymbol{V_0^\pi}$	-6.5%	0.4%	9.3%	-5.9%	1.3%	10.7%	4.6%	15.2%	22.9%

Table 7: Risk measures of the discounted net P&L $\Pi_{0,1}$ for all three considered variations of the modeled hedging program (weekly rebalancing, monthly rebalancing and no hedging at all) and all three variations of the equity-volatility level. Based on 10,000 paths in each corresponding simulation.

Also, in contrast to the change in interest rate levels, the measured hedge effectiveness is noticeably affected by the higher volatility. Using again the results for no hedging as a benchmark and the value-at-risk at a confidence level of 99.5%, the hedge effectiveness decreases from 93% to 89% for the weekly hedge and from 87% to 80% for the monthly hedge if the higher volatility environment is considered. On the other hand, the effectiveness only slightly increases in the reduced volatility environment (to 94% for the weekly hedge and to 89% for the monthly hedge).

If no hedging is considered, the risk measures at a confidence level of 99.5% (as well as those at the 90% level) actually decrease with higher volatility, meaning that, after the increase of the value of liabilities from -0.8% to 7.6% (see Table 5) and – if unhedged – a potentially significant loss for the insurer, the P&L in the subsequent year has a less pronounced downside risk as in the base case. Therefore, in comparison to a stand-alone comparison of the change in the value of liabilities, the overall impact on the insurer is reduced, with the sum of $\mathrm{VaR}_{99.5\%}$ and V_0^π increasing only by 7.7 percentage points from 15.2% to 22.9%.

4 Conclusion

After inception of a variable annuity, the provider usually aims to compensate changes in the value of liabilities by means of a hedging program. However, even if a well-functioning hedging program is in place that is able to effectively replicate (most) changes in the value of liabilities, risk-based capital requirements can still fluctuate due to their dependence on market parameters. As a result, variable annuity providers face the risk of increases in their risk-based capital requirements and, thus, the need for capital injections – even without pricing errors or malfunctioning of the hedging program.

By means of a simulation study, we assessed the effectiveness of a stylized hedging program with different rebalancing frequencies over a one-year time horizon and analyzed the distribution of the insurer's resulting P&L. Using this P&L, we computed different risk measures as indicators for risk-based capital requirements. The resulting capital requirements highly depend on the degree to which the risk-mitigating effect of the hedging program is allowed for in the calculation, with our results showing an increase by a factor of more than ten if no allowance of the risk-mitigating effect is made. The approach we used is comparable to an internal model type approach under Solvency II and, thus, our results can be used as indicator for risk-based capital requirements (with respect to market risk) a variable annuity provider needs to meet in the EU (cf. Section 3.3).

We analyzed how hedge effectiveness and the considered indicators for capital requirements change with different assumptions regarding the interest-rate and equity-volatility environment (cf. Section 3.4). We found that, additionally to the potentially unhedged changes of the value of liabilities that such a change in the market environment causes, the changed parameters also have a considerable impact on risk measures, meaning that in these cases, the insurer faces two stresses at once: the change in the value of liabilities (this might be hedged) and the change in capital requirements (likely unhedged). We also found that, while the impact of the level of interest rates on the effectiveness of the modeled hedging program is rather low, a higher volatility level has a distinct adverse effect on the hedge effectiveness, leading to a further increase of risk-based capital requirements. However, there are also cases where an increase in the value of liabilities was accompanied by a decrease of capital requirements, reducing the overall impact on the insurer. This is the case for some risk measures if no allowance of the hedging program is made and equity volatility is increased.

In conclusion, if the insurer assesses its risk situation with regard to its (new) variable annuity business, it should also incorporate an analysis of future capital requirements, as those may pose an economic risk and can potentially reduce the profitability of the insurer's variable annuity business. Under Solvency II, for instance, such analyses are mandatory in the context of the Own Risk and Solvency Assessment (ORSA). Furthermore, as the sensitivity of capital requirements to market parameters is not easily assessable, thorough numerical analyses appear necessary for a proper assessment of this risk. In such analyses, also the effect of a potentially reduced hedge performance in adverse market environments and a reduced level of recognition of the hedging program's risk-mitigating effect should be considered, as this may lead to additional increases of the capital requirements.

Regarding future research, it seems worthwhile to investigate ways to incorporate changes of risk-based capital requirements into the value that is being replicated by the hedging program, as well as ways to adequately account for future capital requirements in the profit testing of variable annuity products. Also, an analysis that extends the stand-alone analysis of a homogeneous pool of policies (as presented in this paper) to a model with different lines of businesses and heterogeneous pools of policies appears promising. The same applies to a potential analysis that incorporates surrender and/or biometric risk factors (for instance longevity risk), as well as additional market risk factors as, for instance, the level of interest-rate volatility.

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