The Volatility of Mortality

Daniel Bauer
Matthias Börger
Jochen Ruß
Hans-Joachim Zwiesler

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Helmholtzstraße 22
D-89081 Ulm
phone +49 (0) 731/50-31230
fax +49 (0) 731/50-31239
email ifa@ifa-ulm.de

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Agenda

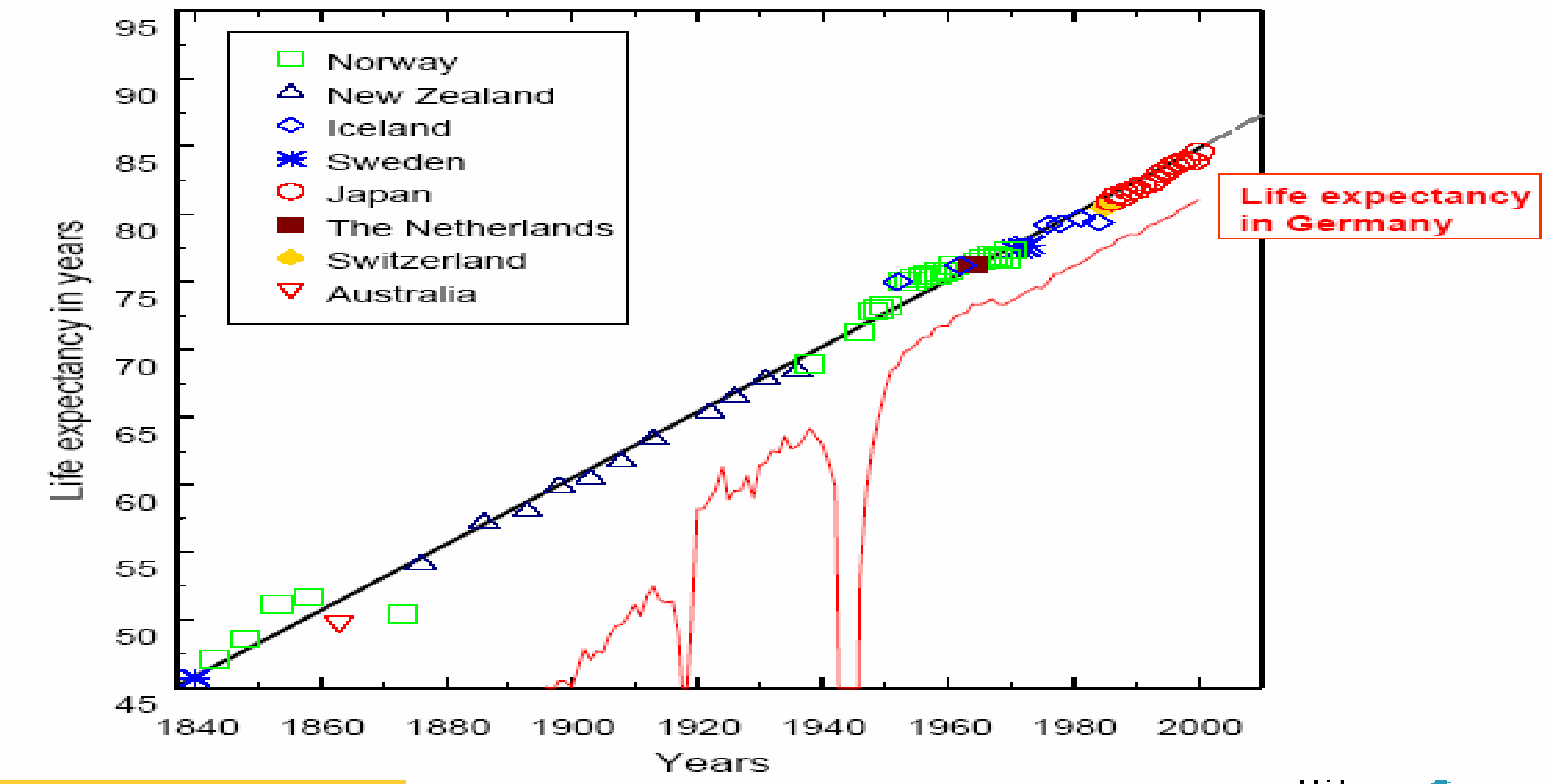
- Introduction
- Theoretical Framework
- Epidemiological Insights
- Specification of the Model
- Calibration
- Applications
- Summary and Outlook

Introduction

- **Traditional actuarial mathematics**
  - Future mortality probabilities are known
  - → Mortality and longevity risk diversifiable
- **Several recent studies**
  - Systematic mortality risk is an important risk factor for life and annuity insurers as well as pension providers
- **In particular, for annuity and pension providers**
  - Longevity risk
  - In many countries: very large longevity exposure in occupational pensions
  - Many countries provide tax incentives in case of annuitization
    → Longevity risk in the insurance industry will increase

Introduction

- **My favorite pictures**
  - 1. Increase in world-wide life expectancy is no temporary trend



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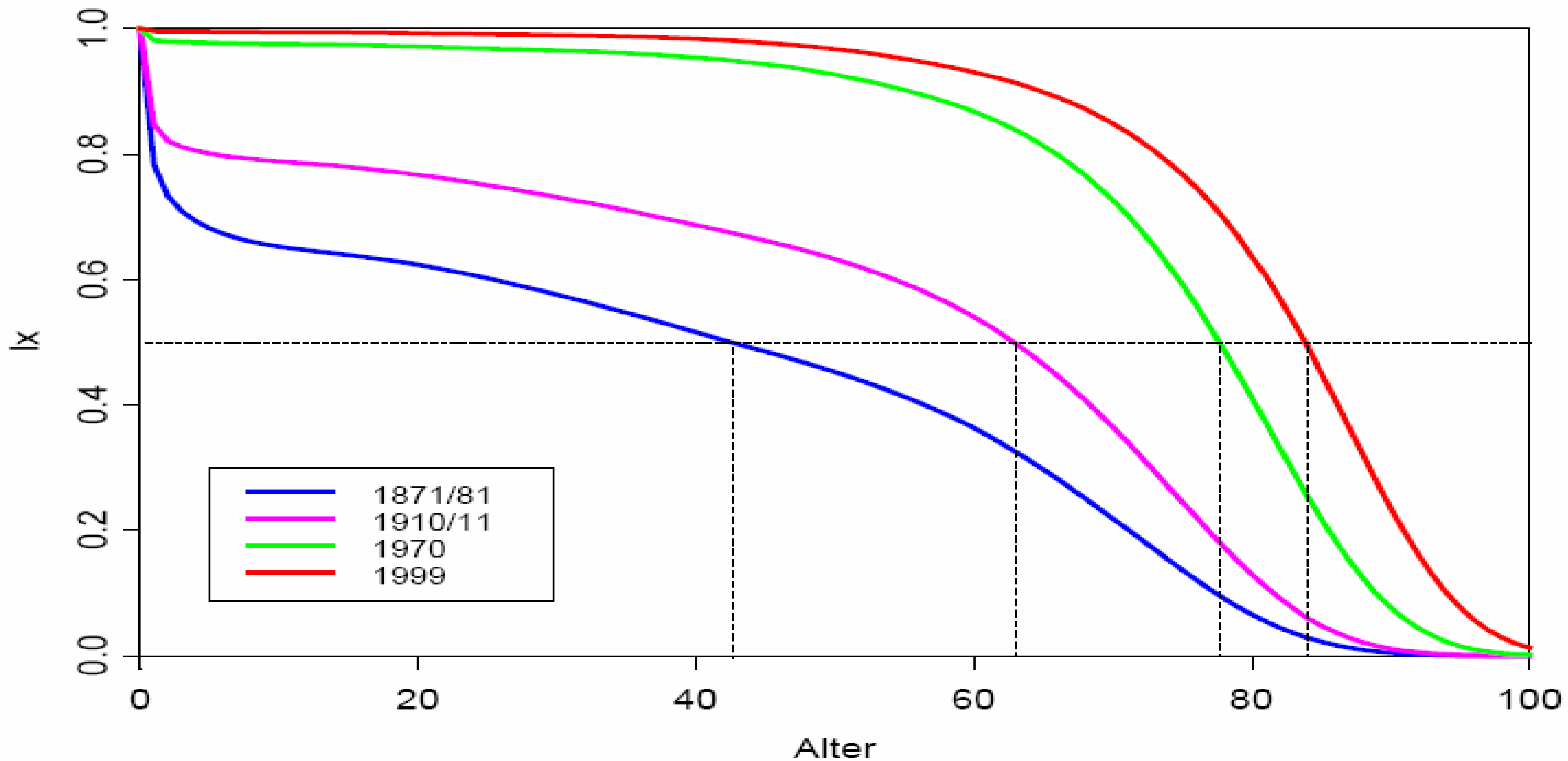
Introduction

- **My favorite pictures**
  - 2. Increase is not “uniform over the mortality curve”



Introduction

- **Since systematic mortality and longevity risk is not diversifiable, other methods of managing this risk have to be applied**
  - e.g. securitization (Cowley and Cummins (2005), Blake et al. (2006))
- **Target: quantify systematic mortality risk and model mortality linked securities**
  - e.g. longevity or survivor bonds
- **Required: Models which consider the stochastic characteristics of the mortality evolution.**
  - In recent literature, a number of such models have been presented
  - Overview in Cairns et al. (2005)

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Introduction

- **Most of the models proposed so far are spot-force models**
  - Simplified: Stochastic versions of period life tables
- **In classical actuarial applications, annuities are usually priced based on generational tables**
  - → Trend is already embedded in the tables
  - Stochastic version of these: Forward mortality models
  - Model for the whole term structure of mortality
  - e.g. Cairns et al. (2005), Miltersen and Persson (2005)
- **However, no concrete forward models have been presented yet**
  - possible exception: stochastic extension of Smith/Olivieri model (Cairns)
- **In what follows: Presentation of two concrete models**

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Theoretical Framework

- **In this presentation we focus on the main ideas and concepts and our results**
- **For more details on theory and formulae we refer to the full paper**
  - available at www.mortalityrisk.org

- **We use mortality bonds as a basis…**
  - bonds with a payoff depending on the portion of a reference population of a certain age, that is still alive at some future point in time
- **… and assume that the forward force of mortality follows**

$$d\tilde{\mu}_t(T, x_0) = \tilde{\alpha}(t, T, x_0)\, dt + \tilde{\sigma}(t, T, x_0)\, d\tilde{W}_t,\; \tilde{\mu}_0(T, x_0) > 0$$

for a finite dimensional Brownian Motion $\tilde{W}$

Theoretical Framework

- **To specify a concrete model, we need**
  - initial condition
  - drift term
  - volatility structure
- **Heath-Jarrow-Morton condition (HJM): Under the risk neutral measure Q, the volatility structure specifies the drift term**
  - → initial condition and volatility structure sufficient
- **Approach for calibrating volatility under Q (proposed by Bauer and Russ, 2006)**
  - Calibrate to prices of embedded guarantees in Variable Annuities with Guaranteed Minimum Benefits
  - However: Market is not (yet) sufficiently liquid
  - Therefore, in what follows, we work with observed mortality data, i.e. under the real-world measure P

Theoretical Framework

- **Some results under P**
  - We show that there is a P-HJM condition similar to the Q-HJM condition
  - A model calibrated under P can be used to determine expected payoffs, percentiles, etc. of mortality contingent claims
  - However, it is not immediately suitable as a pricing model
  - Under the assumption of mortality risk-neutrality, the models coincide
  - In general, in order to specify a corresponding pricing model, the market price of risk has to be specified
  - In the Gaussian case and under the assumption of a deterministic market price of risk, the volatilities of the P and Q models coincide
    - The initial risk-neutral forward plane, the Q-HJM condition, and a P-calibrated volatility structure yield a pricing model

Theoretical Framework

- **The resulting pricing model can be used to**
  - determine the values of mortality securitization products, e.g., longevity bonds
  - determine values of mortality contingent options within insurance products such as Guaranteed Annuity Options or Guaranteed Minimum Benefits within Variable Annuities
  - etc.

- **Still open: Where do we find a suitable volatility structure under P?**

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Epidemiological Insights

- **Since mortality data are not collected frequently (unlike e.g. interest rate data), standard statistical methods like principal component analysis are not possible**
- **Furthermore, the question how mortality might evolve is not primarily a statistical / mathematical / actuarial question**
- **Thus, we tried to derive a suitable model structure from epidemiological / medical insights**

- **However, "typical" epidemiological approaches are not suitable in our framework**
  - Cause of death models or risk factor models
- **The resulting number of parameters would make the model unfeasible**

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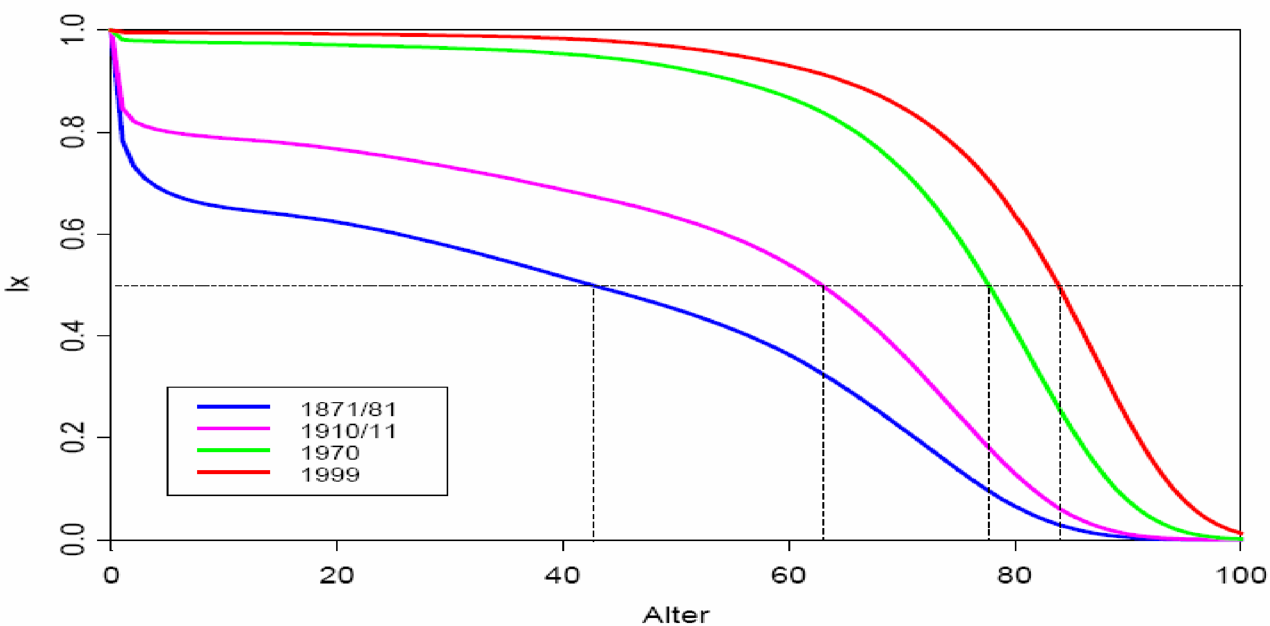
Epidemiological Insights

- **Idea: Combine effects that affect mortality for similar ages at similar points in time**

- **Qualitative epidemiological insights – age and term**
  - Strong changes in mortality happen for different ages at different times



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Epidemiological Insights

- **Idea: Combine effects that affect mortality for similar ages at similar points in time**

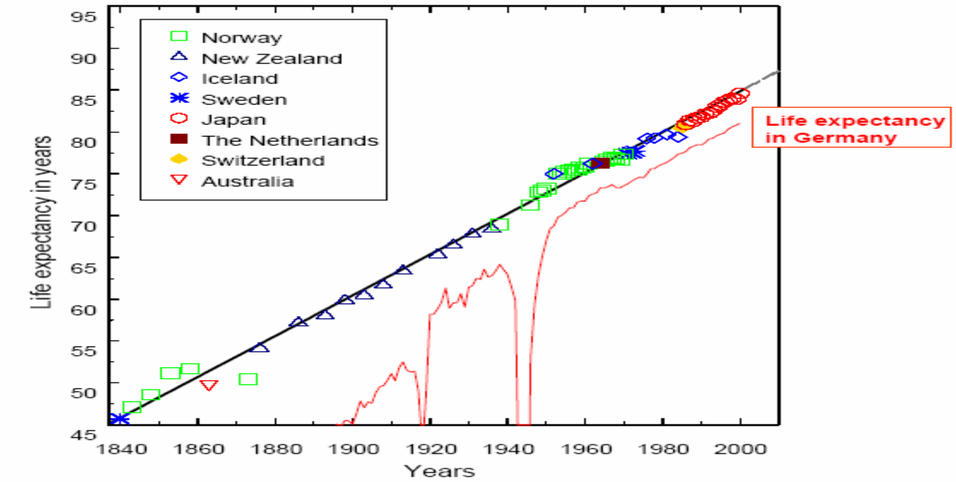
- **Qualitative epidemiological insights – age and term**
  - Strong changes in mortality happen for different ages at different times
    - → Grouping by age and time appears reasonable
    - → Use three age groups: young, mid, old
  - Old age starts at 80
    - Epidemiological studies show that currently the strongest improve in longevity happens at these ages
    - Although, e.g. ages 70+ are usually referred to as old, from a "mortality point of view" they are more similar to mid than old ages (Wiesner (2001), Boleslawski and Tabeau (2001))
  - Other groups: Young = age 20 to 55; mid = 55-80

Epidemiological Insights

- **Qualitative epidemiological insights – age and term**
  - We need a short term a mid term and a long term effect
    - Short-term effect affects only the "near future"
      - wave of influenza, bird flu,…
      - hits all ages "similarly" → not age dependent
  - One mid-term effect for each age group
  - One long term effect → not age dependent
    - When long term effect kicks in, all cohorts considered are rather old already
  - Finally, we introduce one overall effect (see below)
  - → 6 effects

Epidemiological Insights

- **Qualitative epidemiological insights – Distributional properties**
  - No limit to a potential temporary increase in mortality
  - Controversial discussion whether or not an upper limit exists for human life expectancy (decrease in mortality)
    - Some favor an upper limit
    - Other favor an "increasing boundary" which for our purpose can be treated as a "time dependent upper limit"



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Epidemiological Insights

- **Qualitative epidemiological insights – Distributional properties**
  - No limit to a potential temporary increase in mortality
  - Controversial discussion whether or not an upper limit exists for human life expectancy (decrease in mortality)
    - Some favor an upper limit
    - Other favor an "increasing boundary" which for our purpose can be treated as a "time dependent upper limit"
  - → Forward force of mortality should have a positively skewed distribution with rather light tails.

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Specification of the model

- **The 6 effects**
  - exponential building blocks
  - The age dependent mid-term effect: bell shaped with center at the middle of the age range

general:	$\sigma_1(t,T,x_0) = c_1$
short-term:	$\sigma_2(t,T,x_0) = c_2 \cdot \exp\left(-w_2(T-t)\right)$
young age:	$\sigma_3(t,T,x_0) = c_3 \cdot \exp\left(-w_{31}(T-t-20)^2 - w_{32}(x_0+T-37.5)^2\right)$
middle age:	$\sigma_4(t,T,x_0) = c_4 \cdot \exp\left(-w_{41}(T-t-20)^2 - w_{42}(x_0+T-67.5)^2\right)$
old age:	$\sigma_5(t,T,x_0) = c_5 \cdot \exp\left(-w_{51}(T-t-20)^2 - w_{52}(x_0+T-110)^2\right)$
long-term:	$\sigma_6(t,T,x_0) = c_6 \cdot \exp\left(-w_6(T-t-120)^2\right)$

- **→ 14 parameters – that's a lot!**

Specification of the model

- **Eliminating parameters for the sake of feasibility**
  - Short term effect: After one year reduced to 10% → $w_2=\log(1/10)$
  - "Smooth transition" from one effect to the next
    - with respect to age and time
    - Each effect is reduced to 50% at the boundary to the next effect
      - Rather arbitrary – other restrictions yield different models
    - This transition and the overall effect, allow for correlation

  - → Resulting (not yet final) model has 6 parameters

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Specification of the model

- **Final adjustment: Distributional properties**
  - Model so far is Gaussian
  - Changes in forward mortality are absolute, i.e. volatility is decreasing in age
  - Moving to “$d\mu/\mu$” would solve that but result in a log-normal-type distribution which is not desired
  - We therefore propose to use correction terms
    - multiply volatility derived so far with a suitable correction term

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Specification of the model

- **Model 1: Gompertz correction term → Gaussian model**
  - Assuming the Gompertz curve to be a reasonable approximation for the force of mortality μ, σ is roughly proportional to μ; model remains Gaussian

general:	$\sigma_1(t,T,x_0) = c_1 \cdot \exp\left(a(x_0+T)+b\right)$
short-term:	$\sigma_2(t,T,x_0) = c_2 \cdot \exp\left(a(x_0+T)+b\right) \cdot \exp\left(\log(0.1)(T-t)\right)$
young age:	$\sigma_3(t,T,x_0) = c_3 \cdot \exp\left(a(x_0+T)+b\right) \cdot \exp\left(\frac{\log(0.5)}{20^2}(T-t-20)^2 + \frac{\log(0.5)}{17.5^2}(x_0+T-37.5)^2\right)$
middle age:	$\sigma_4(t,T,x_0) = c_4 \cdot \exp\left(a(x_0+T)+b\right) \cdot \exp\left(\frac{\log(0.5)}{20^2}(T-t-20)^2 + \frac{\log(0.5)}{12.5^2}(x_0+T-67.5)^2\right)$
old age:	$\sigma_5(t,T,x_0) = c_5 \cdot \exp\left(a(x_0+T)+b\right) \cdot \exp\left(\frac{\log(0.5)}{20^2}(T-t-20)^2 + \frac{\log(0.5)}{30^2}(x_0+T-110)^2\right)$
long-term:	$\sigma_6(t,T,x_0) = c_6 \cdot \exp\left(a(x_0+T)+b\right) \cdot \exp\left(\frac{\log(0.5)}{80^2}(T-t-120)^2\right)$

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Specification of the model

- **Model 2: correction term = SQRT(μ) → general (or Square-root) model**
  - square-root Feller diffusion process
  - roughly X^2 distributed (cf. Glasserman (2004))
  - positively skewed and light tailed as desired

general:	$\sigma_1(t,T,x_0) = c_1 \cdot \sqrt{\hat{\mu}_t(T,x_0)}$
short-term:	$\sigma_2(t,T,x_0) = c_2 \cdot \sqrt{\hat{\mu}_t(T,x_0)} \cdot \exp\left(\log(0.1)(T-t)\right)$
young age:	$\sigma_3(t,T,x_0) = c_3 \cdot \sqrt{\hat{\mu}_t(T,x_0)} \cdot \exp\left(\frac{\log(0.5)}{20^2}(T-t-20)^2 + \frac{\log(0.5)}{17.5^2}(x_0+T-37.5)^2\right)$
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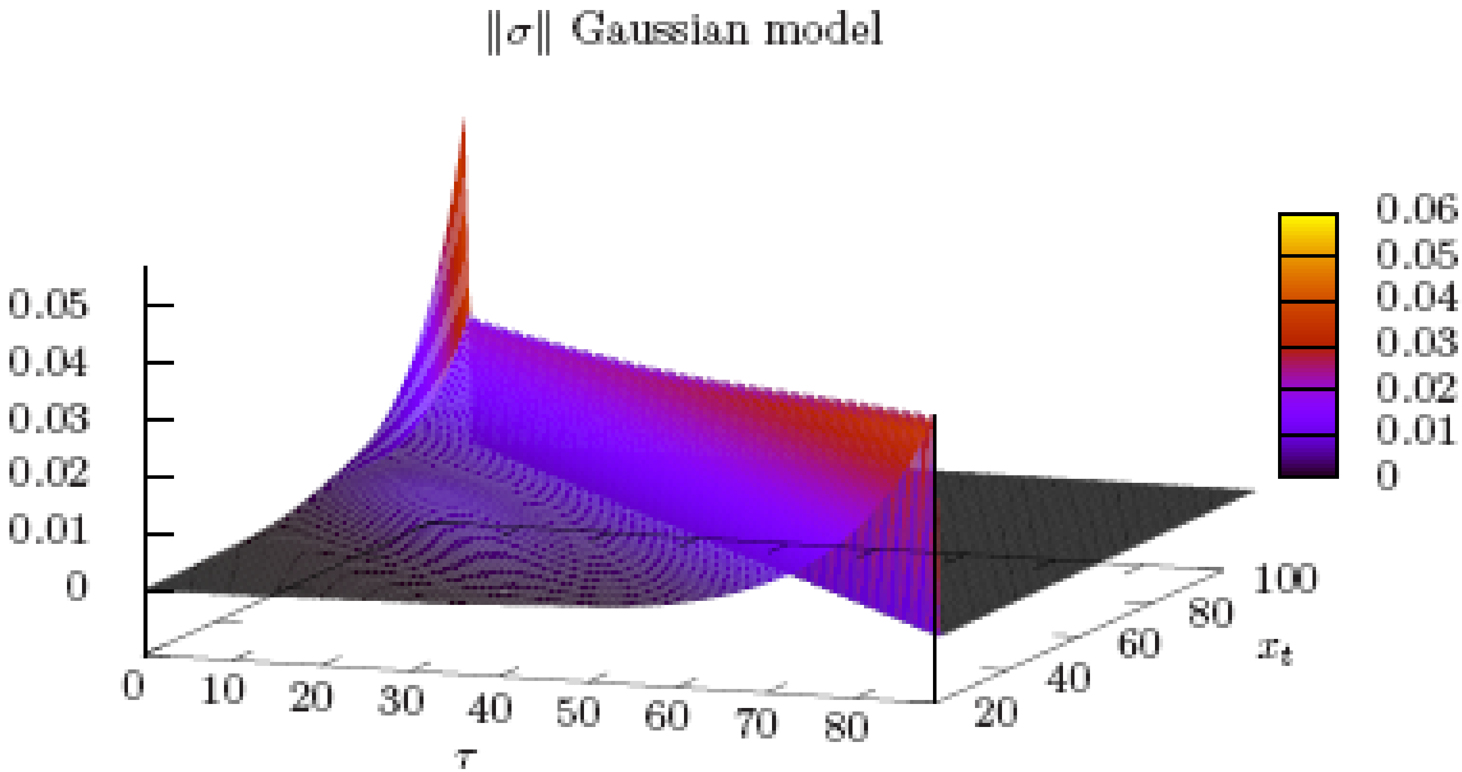
Calibration

- **In the Gaussian case, we find close form solutions** (up to the numerical solution of multidimensional integrals) **for the moments of the distribution of the difference of the logs of the change in survival probabilities over time**
  - → Thus we know the probability density
  - This can be used in a maximum likelihood framework to perform a calibration
- **For the general model, we develop a pseudo-maximum likelihood methodology (based on Monte Carlo simulation)**
  - numerically complex
  - details see paper
- **We calibrate our models to the Group Annuity Mortality Tables (GAM tables) published by the Society of Actuaries**
  - 1951, 1971, 1983 and 1994 generation table
  - "artificial" creation of larger dataset seems possible

Calibration

- **calibration results for the Gaussian model (absolute)**
  - norm of $\sigma(t,T,x_0)$; $x_t := x_0 + t$ and $\tau = T-t$



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Calibration

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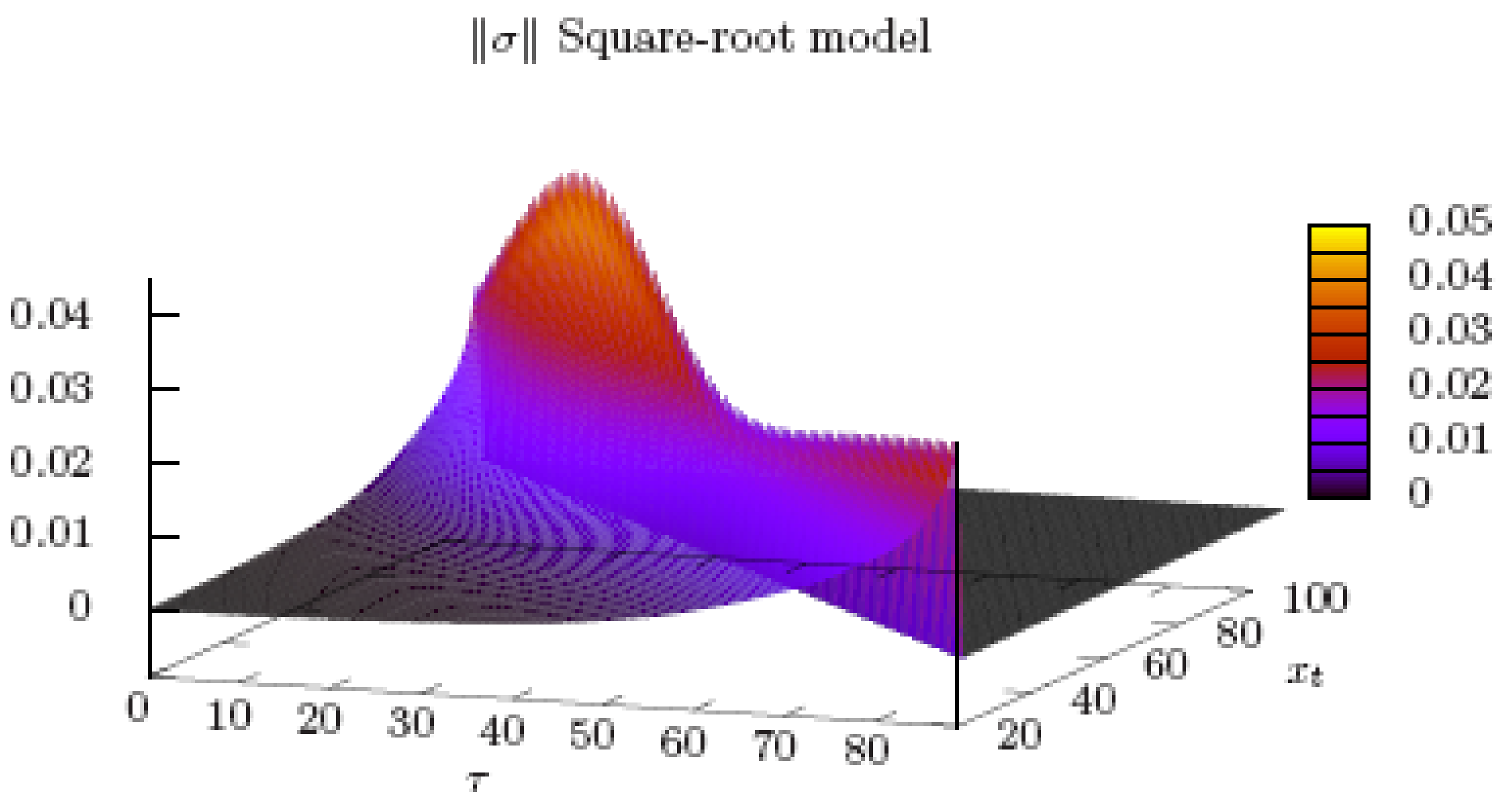

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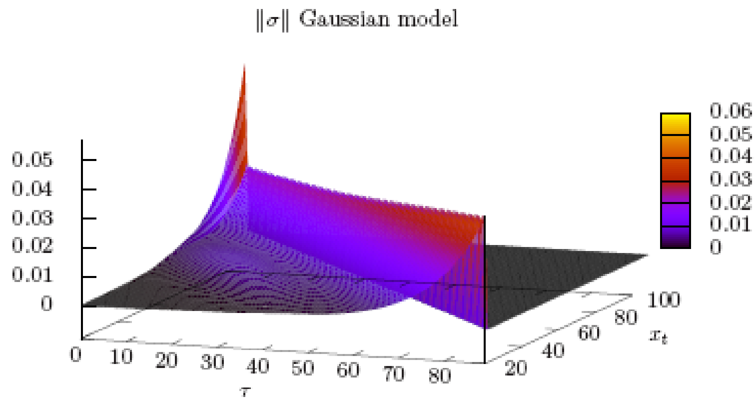
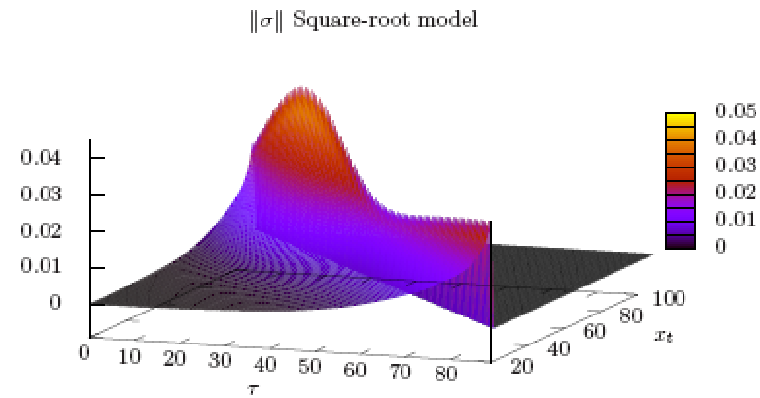
Calibration

- **calibration results**

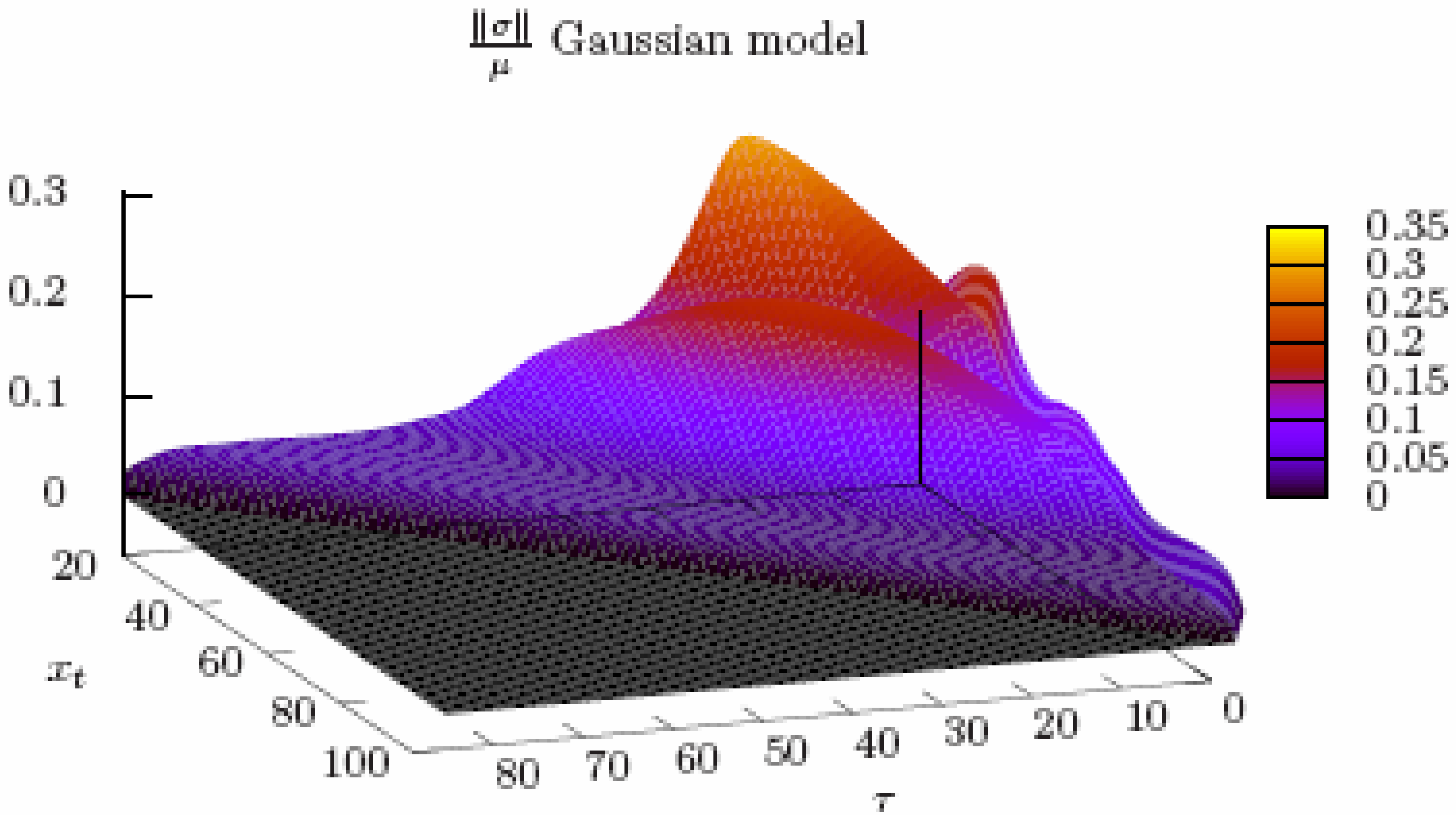




- **exponential trend in τ for each x_t and in x_t for each τ**
- **volatilities for the same age at maturity increase in the time to maturity → the further in the future, the more uncertainty**
- **short term effect more pronounced in the Gaussian model; mid-term effect larger for the Square-root model**
  - The comparison is somewhat misleading (different weightings)
  - → also look at relative values

Calibration

- **calibration results for the Gaussian model (relative)**



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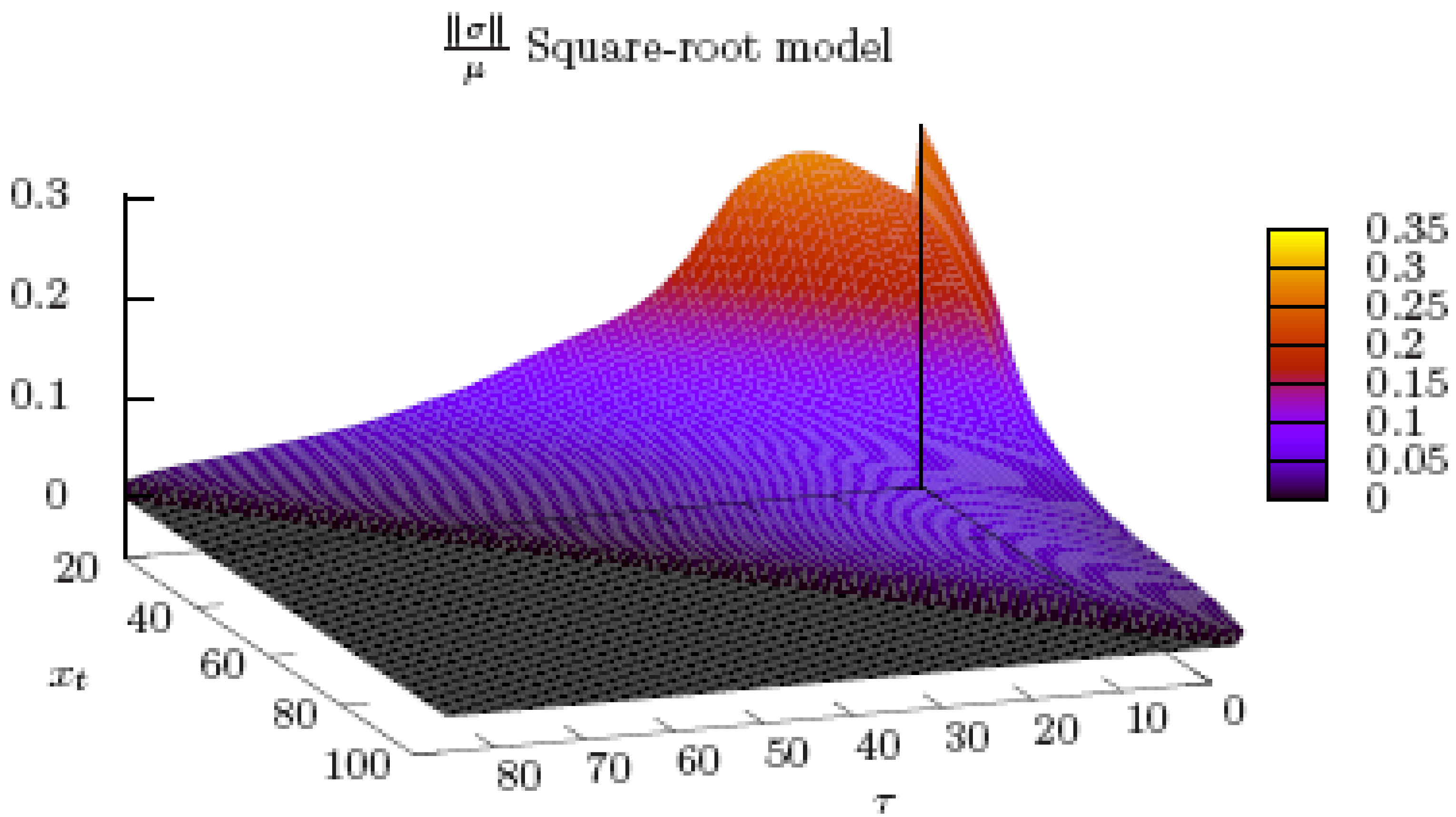
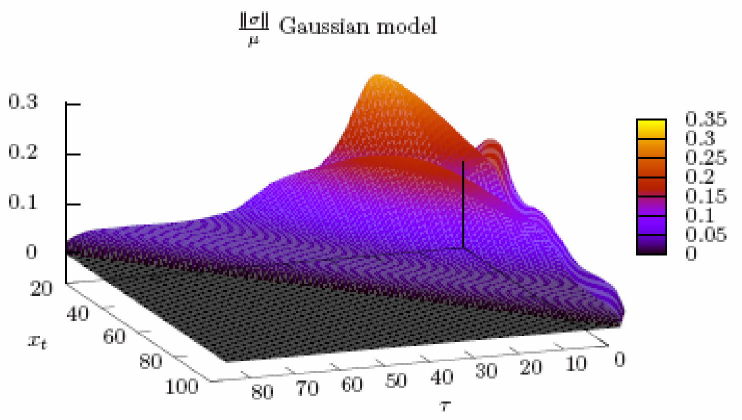
Calibration

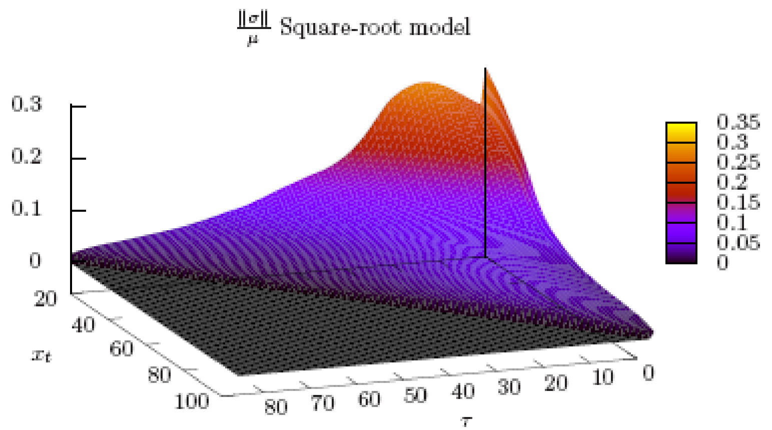
- **calibration results for the general model (relative)**



Calibration

- **calibration results**





- **short-term effect is now more pronounced in the Square-root model**
- **young age effect is very large in both models**
- **mid age effect is hardly noticeable for the Square-root case but quite pronounced for the Gaussian calibration.**
- **overall relative volatility decreases in the age at maturity**
  - this is different for the absolute volatility
- **general structure and size of the volatilities are fairly similar**

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Applications

- **In the paper, we discuss several areas of application**
- **One example:**
  - Option with payoff $C_T = \left({}_Tp_{x_0}^{(T)} - K \right)^+$,

    with strike $K := (1+\alpha) E_P \left[{}_Tp_{x_0}^{(T)} \right]$

- **Trigger probabilities and expected payoff under P (T=20, α=2.5)**

Model	*Gaussian model*		*Square-root model*	
	$P\left({}_Tp_{x_0}^{(T)} > K\right)$	$E\left[\left({}_Tp_{x_0}^{(T)} - K\right)^+\right]$	$P\left({}_Tp_{x_0}^{(T)} > K\right)$	$E\left[\left({}_Tp_{x_0}^{(T)} - K\right)^+\right]$
$x_0 = 65$	0.4076	0.0491	0.4214	0.0200
$x_0 = 70$	0.4021	0.0334	0.4268	0.0194
$x_0 = 75$	0.3877	0.0174	0.4190	0.0122
$x_0 = 80$	0.3567	0.0058	0.3961	0.0042
$x_0 = 85$	0.3053	0.0011	0.3670	0.0008

Applications

- **Trigger probabilities slightly higher in the Square-root model**
  - consequence of the distributional differences of the models
    - Gaussian: mean and median coincide
    - Square-root: mean exceeds the median
      → higher probability for intensities just below the expected value
      → increase of the probability of survival probabilities slightly above the expected value

- **Expected payoff considerably larger in the Gaussian model**
  - relative difference between the expected payoffs increases in age
  - consequence of the far more pronounced middle age effect in the Gaussian model

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Applications

- **Using the market price of risk obtained by Lin and Cox (2005), we also priced this mortality option**

	$p(0,T)\,E_P\left[{}_Tp_{x_0}^{(T)}\right]$	$p(0,T)\,E_P\left[\left({}_Tp_{x_0}^{(T)} - K\right)^+\right]$	$p(0,T)\,E_Q\left[\left({}_Tp_{x_0}^{(T)} - K\right)^+\right]$
$x_0 = 65$	0.1006	0.0112	0.0214
$x_0 = 70$	0.058	0.0076	0.0168
$x_0 = 75$	0.0228	0.0040	0.0095
$x_0 = 80$	0.0053	0.0013	0.0033
$x_0 = 85$	0.0007	0.0003	0.0006

- **Price is rather high**
  - Insurer might combine this "long call on the survival probability" with a short put or chose a different strike.

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Summary

- **Whilst there seems to be consent that forward-type models are appropriate to assess stochastic mortality, so far no concrete model has been proposed**
  - Reasons: lack of data, in particular price data; medical questions
- **Therefore, we chose a different approach**
  - model with age and time effects
  - model structure based on epidemiological insight
  - model calibration based on observed mortality data (under P)
- **We derived several new theoretical results about the relation between P and Q models**
- **We specified two concrete models**
- **Model calibration**
  - semi-analytical for Gaussian model
  - new quasi maximum likelihood method developed for the general model
- **We showed several applications of the model**

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Limitations and Outlook

- **Certain aspects of the approach are rather arbitrary**
  - age limits; exponential functions; parameter reduction; correction terms
  - other choices would lead to different results
  - empirical investigations based on “synthetic” data might yield further insights regarding the adequacy of the approach
- **Calibration Procedures:**
  - Problem: Data not necessarily coherent with possible structure implied by the model
    - → Gaussian case: Numerical instabilities due to over-determinedness of linear system of equation; only possible to incorporate “few” values leading to certain arbitrariness of the choice
    - → General case: Certain arbitrariness in the definition of some “acceptance level” within the used distance function
  - Solution: Smoothing the residuals – What smoothing function to use?
    - → “Consistency” Problem

Thank you very much!

Helmholtzstraße 22
D-89081 Ulm
phone +49 (0) 731/50-31230
fax +49 (0) 731/50-31239
email ifa@ifa-ulm.de