Solvency II and Nested Simulations – a Least-Squares Monte Carlo Approach

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Motivation

- Solvency II: New regulatory framework for insurance companies in the European Union
- Key aspect: Determine required risk capital (SCR) for a one-year time horizon based on a market-consistent valuation of assets and liabilities
- Standard model: Approximation of SCR via square-root formula
  $\Rightarrow$ Various deficiencies (cf. Pfeifer/Strassburger (2008), Sandström (2007)).
- Alternative: Multivariate approach based on stochastic model for the insurance company (Internal Model).
- Problems:
  - Valuation of life insurance contracts in closed form not possible (due to embedded options and guarantees)
  - Unsolved numerical and computational problems

$\Rightarrow$ This paper provides a mathematical framework for the calculation of the SCR and discusses different approaches for the numerical implementation.

Definitions

Assessment of solvency position can be split into two components:

1. Available Capital (AC_0)
   - Amount of financial resources available at time $t = 0$ which can serve as a buffer against risks and absorb financial losses.

Market consistent valuation of assets and liabilities

$$AC_0 := MVA_0 - MVL_0 = MCEV_0$$

- $MCEV_0$ denotes the market consistent embedded value, i.e. $MCEV_0 = ANAV_0 + PVPF_0 - CoC_0$, where
  - $ANAV_0$ is derived from statutory shareholders' equity,
  - $PVFP_0$ is the present value of past-taxation shareholder cash flows from the assets backing (statutory) liabilities and
  - CoC_0 is the Cost-of-Capital charge (not discussed further here).

- Main computational issue: calculation of $PVFP_0$.

Definitions

Assessment of solvency position can be split into two components:

2. Solvency Capital Requirement (SCR)
   - SCR is based on the Available Capital at $t = 1$, where $AC_1 := MCEV_1 + X_1$ and X_1 denotes shareholder cash flows at $t = 1$.
   - Intuition: An insurance company is considered solvent under Solvency II if its Available Capital at $t = 1$ is positive with a probability of at least $\alpha = 99.5\%$.
   - Therefore consider loss function $L := AC_0 - AC_1/(1+i)$ where i denotes the one-year risk-free rate at $t = 0$.

SCR definition

$$SCR := \text{argmin}_x \left\{ P\left(AC_0 - AC_1/(1+i) > x\right) \le 0.5\% \right\} = VaR_{99.5\%}(L)$$

- Main computational issue: calculation of 99.5%-quantile of $-AC_1$.

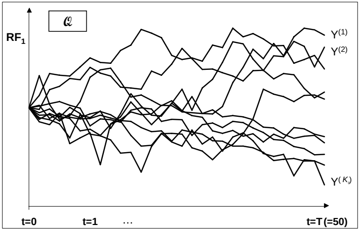
Mathematical Framework

- Complete filtered probability space $\left(\Omega, \mathcal{F}, \mathcal{P}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}\right)$
  - $\mathcal{P}$ so-called *real-world (physical) measure*
  - *Risk-neutral measure* $\mathcal{Q}$ equivalent to $\mathcal{P}$
- State process: $(Y_t)_{t \in [0,T]} = \left(Y_t^{(1)}, \ldots, Y_t^{(d)}\right)_{t \in [0,T]}$ of sufficiently regular Markov processes that describes the stochasticity of the market
  - Numéraire process: $B_t = \exp\left(\int_0^t r_u du\right)$, $r_t = r(t, Y_t)$
- Cash flow projection model, i.e. the future profits of the insurance company X_t $(t = 1, \ldots, T)$ can be described as

$$X_t = f_t(Y_s, s \in [0, t])$$

Valuation at $t = 0$

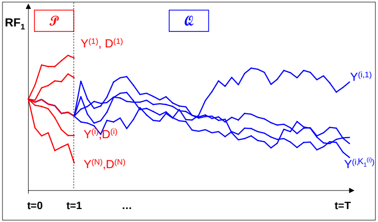
- Target: $AC_0 = \underbrace{ANAV_0}_{\text{from statutory balance sheet}} + \underbrace{\mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} \exp(-\int_0^t r_u\,du) X_t\right]}_{=:V_0}$
- Problem: No closed form solution for V_0



- Monte Carlo simulations: $\widetilde{V}_0(K_0) = \frac{1}{K_0} \sum_{k=1}^{K_0} \sum_{t=1}^{T} \exp\left(-\int_0^t r_u^{(k)}\,du\right) X_t^{(k)}$

Valuation at $t = 1$

- Target: Distr. of $AC_1 = ANAV_1 + \underbrace{\mathbb{E}^{Q}\left[\sum_{t=2}^{T} \exp(-\int_1^t r_u\, du) X_t \,\middle|\, (Y_1, D_1)\right]}_{=:V_1} + X_1 \sim F$



Simulate	N	first-year paths "under P": $(Y_1^{(i)}, D_1^{(i)})$
Simulate	K_1	paths "under Q" starting in $(Y_1^{(i)}, D_1^{(i)})$: determine $V_1^{(i)}$
	$N \times K_1$	paths

Estimator in the Nested Simulations Approach

Estimated SCR

We now have

1. $\widetilde{\text{AC}}_0(K_0) = \text{ANAV}_0 + \widetilde{V}_0(K_0)$
2. $\widetilde{\text{AC}}_1^{(i)}(K_1) := \text{ANAV}_1^{(i)} + \widetilde{V}_1^{(i)}(K_1) + X_1^{(i)}, 1 \leq i \leq N.$

Hence, we can estimate SCR by

$$\widetilde{\text{SCR}} = \widetilde{\text{AC}}_0 + \frac{\widetilde{z_{(m)}}}{1+i}$$

where $\widetilde{z_{(m)}}$ is the m^{th} order statistic of $-\widetilde{\text{AC}}_1^{(i)}$ and $m = \lfloor N \cdot 0.995 + 0.5 \rfloor$.

- Within the estimation process, we have three sources of error:
  1. Estimation of AC_0 with only K_0 sample paths
  2. Estimation of the quantile with only N real-world scenarios
  3. Estimation of $\text{AC}_1^{(i)}$ with only K_1 inner simulations $\forall i$

$\Rightarrow$ Analysis of the resulting error in our estimate $\widetilde{\text{SCR}}$

Variance-Bias Tradeoff – Choice of K_0, N and K_1

- Idea: minimize the Mean-Square Error (MSE)

$$\text{MSE} = \mathbb{E}\left[(\widetilde{\text{SCR}} - \text{SCR})^2\right] = Var(\widetilde{\text{SCR}}) + \Big[\underbrace{\mathbb{E}(\widetilde{\text{SCR}}) - \text{SCR}}_{\text{bias}}\Big]^2$$

- Similar to Gordy/Juneja (2008), we obtain:

Optimization problem in K_0, N and K_1

$$\frac{\sigma_0^2}{K_0} + \frac{\alpha(1-\alpha)}{(N+2)f^2(\text{SCR})} + \frac{\theta_\alpha^2}{K_1^2 \cdot f^2(\text{SCR})} \rightarrow \min$$

subject to the effort restriction $K_0 + N \cdot K_1 = \Gamma$.

- Can be solved using Lagrangian multipliers (for given computational budget Γ).
- Note: bias is positive in practical applications resulting in a **systematic overestimation of the SCR**.
- Problem: To make bias small (for 99.5% confidence level), K_1 may not be chosen "too small" $\rightarrow$ **Immense computational effort!**

Least-Squares-Algorithm

- Based on LSM approach by Longstaff/Schwartz (2001) for the valuation of non-European options (see also Clément et al. (2002)).
- Algorithm:
  - Simulate N scenarios (first year $\mathcal{P}$, other years $\mathcal{Q}$)

$$PV_1^{(i)}(\omega_i) := \sum_{t=2}^{T} \exp\left\{-\int_1^t r_s(\omega_i)\, ds\right\} X_t(\omega_i) = \mathbb{E}^{\mathcal{Q}}\left[PV_1^{(i)} \middle| \mathcal{F}_1 \right] + \varepsilon_i, \qquad 1 \le i \le N$$

  - 1^{st} step: Approximate V_1 by finite sum of appropriate basis functions

$$V_1 = \mathbb{E}^{\mathcal{Q}}\left[\sum_{t=2}^{T} \exp(-\int_1^t r_u\, du) X_t \middle| (Y_1, D_1) \right] \approx \hat{V}_1^{(M)}(Y_1, D_1) = \sum_{k=1}^{M} \alpha_k \cdot e_k(Y_1, D_1)$$

  - 2^{nd} step: Estimate unknown parameter vector α via regression:

$$\hat{\alpha}^{(N)} = \text{argmin}_{\alpha \in \mathbb{R}^M} \left\{ \sum_{i=1}^{N} \left[PV_1^{(i)} - \sum_{k=1}^{M} \alpha_k \cdot e_k\left(Y_1^{(i)}, D_1^{(i)}\right) \right]^2 \right\}$$

  - Estimate Available Capital:

$$\widehat{\text{AC}}_1^{(i)} = \text{ANAV}_1^{(i)} + \sum_{k=1}^{M} \hat{\alpha}_k^{(N)} \cdot e_k(Y_1^{(i)}, D_1^{(i)}) + X_1^{(i)}, \quad 1 \le i \le N$$

Least-Squares-Algorithm: Does it work?

Issues to consider:

- Suitability of regression approach
- Convergence of the algorithm
- Bias (finite number of basis functions, estimation of regression parameters)
- Choice of regression function

$\Rightarrow$ Ultimate test: How well does it perform in a somewhat realistic framework?

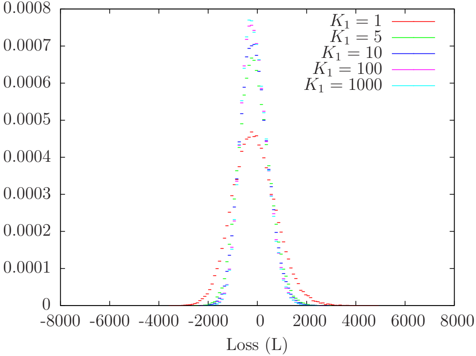
Example: A Participating Life Insurance Contract

- Term-fix insurance contract with minimum interest rate guarantee
- Bonus distribution models obligatory payments to the policyholder (MUST-case from Bauer et al. (2006))
- No mortality $\Rightarrow$ no biometric risk

- Dividends d_t are paid to the shareholders
- Company obtains additional contribution c_t from its shareholders in case of a shortfall

- Asset model: Extended Black-Scholes model with stochastic interest rates (see Bauer/Zaglauer (2008))

Bias in Nested Simulations, $N = 100,000$



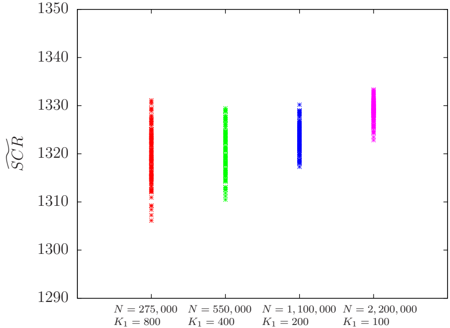
K_1	$\widetilde{SCR}$	$AC_0/\widetilde{SCR}$
1	2,321.0	75%
5	1,538.2	113%
10	1,432.6	121%
100	1,335.3	130%
1,000	1,324.8	131%

- Choice of K_1 significantly affects SCR!
- Estimation of θ_α via pilot simulation with $N = 100,000$, $K_1 = 100$ and regression/finite difference approximation:

  $$\hat{\theta}_\alpha \approx 0.027 \Rightarrow (K_0; N; K_1) = (2,500,000\,; 550,000\,; 400) \text{ approx. optimal}$$

- Calculation takes about 35 minutes.

Comparison of different $(K_0; N; K_1)$ with $\Gamma = 222,500,000$



→ Based on 120 runs of simulations (approx. 35 min each)

N	K_1	Mean ($\widetilde{SCR}$)	Empirical Variance	Estimated Bias	Estimated MSE	Corrected Mean
275,000	800	1319.6	28.0	1.5	30.2	1318.1
550,000	**400**	**1320.5**	**19.3**	**3.0**	**28.2**	**1317.5**
1,100,000	200	1323.1	8.8	5.9	43.9	1317.2
2,200,000	100	1328.9	4.4	11.8	143.2	1317.1

Table: Choice of N and K_1 ($K_0 = 2,500,000$), 120 runs

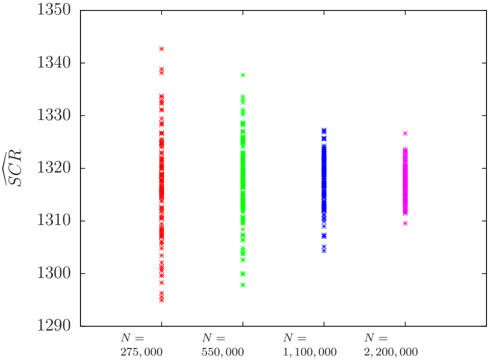
Choice of the Regression Function in the LSM Approach

#	Regression Function	Mean $(\widehat{SCR})$
1	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1$	921.1
2	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2$	1141.9
3	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1$	1309.2
4	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2$	1330.1
5	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1$	1297.5
6	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1$	1302.5
7	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1 + \hat{\alpha}_7^{(N)} \cdot A_1 \cdot e^{r_1}$	1309.2
8	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1 + \hat{\alpha}_7^{(N)} \cdot A_1 \cdot e^{r_1}$ $+\hat{\alpha}_8^{(N)} \cdot L_1 \cdot e^{r_1}$	1316.5
9	$\hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} \cdot A_1 + \hat{\alpha}_2^{(N)} \cdot A_1^2 + \hat{\alpha}_3^{(N)} \cdot r_1 + \hat{\alpha}_4^{(N)} \cdot r_1^2 + \hat{\alpha}_5^{(N)} \cdot L_1 + \hat{\alpha}_6^{(N)} \cdot x_1 + \hat{\alpha}_7^{(N)} \cdot A_1 \cdot e^{r_1}$ $+\hat{\alpha}_8^{(N)} \cdot L_1 \cdot e^{r_1} + \hat{\alpha}_9^{(N)} \cdot e^{A_1/10000}$	**1317.5**

Table: Estimated SCR for different choices of the regression function, $N = 550,000$

- Influence of basis function is quite pronounced.
- For "good" choices, the estimated SCR is close to the result obtained via Nested Simulations.
- "Good" choices appear to remain "good" for different parameters.
- Calculation takes only about 30 seconds.

Comparison of different *N* in the LSM Approach



N	Mean ($\widehat{SCR}$)	Empirical Variance	Solvency Ratio
275,000	1316.9	87.5	132%
550,000	**1317.5**	**62.6**	**132%**
1,100,000	1317.4	23.5	132%
2,200,000	1317.2	10.5	132%

Table: Results for the LSM estimator, 120 runs

Summary

- Nested Simulations:
  - $\longrightarrow$ Inadequate choice of (K_0, N, K_1) in nested simulations may yield erroneous outcomes.
  - $\longrightarrow$ Immense computational effort to achieve accurate results.
- LSM:
  - $\longrightarrow$ Fast approach to achieve relatively accurate results.
  - $\longrightarrow$ Results are similarly positive when calculating SCR for longer time horizons ("richer sigma field").
  - $\longrightarrow$ Care is required in choice of regression function even though simple algorithms yield good results in our applications.
  - $\longrightarrow$ Open question: theoretical results regarding validity of approximation.

Future Research

- Improvement of the Nested Simulations Approach by variance reduction techniques, QMC and screening procedures.
- Use of statistical methods to determine the regression function.
- Analysis of other risk measures, such as TVaR.

Literature

- D. Bauer, R. Kiesel, A. Kling, and J. Ruß: Risk-neutral valuation of participating life insurance contracts. *Insurance: Mathematics and Economics*, 39:171–183, 2006.
- E. Clément, D. Lamberton, and P. Protter: An analysis of a least squares regression method for American option pricing. *Finance and Stochastics*, 6:449–471, 2002.
- M.B. Gordy and S. Juneja: Nested simulations in portfolio risk measurement, 2008. Finance and Economics Discussion Series, submitted to Management Science.
- F.A. Longstaff and E.S. Schwartz: Valuing American options by simulation: A simple least-squares approach. *The Review of Financial Studies*, 14:113–147, 2001.
- D. Pfeifer and D. Strassburger: Solvency II: stability problems with the SCR aggregation formula. *Scandinavian Actuarial Journal*, 1:61–77, 2008.
- A. Sandström: Solvency II: Calibration for skewness. *Scandinavian Actuarial Journal*, 2:126–134, 2007.
- K. Zaglauer and D. Bauer: Risk-neutral valuation of participating life insurance contracts in a stochastic interest rate environment. *Insurance: Mathematics and Economics*, 43:29–40, 2008.

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Thanks for your attention!