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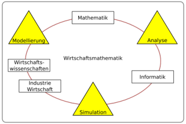
Decomposition of life insurance liabilities into risk factors – theory and application to annuity conversion options

Joint work with Daniel Bauer, Marcus C. Christiansen, Alexander Kling



Research Training Group 1100

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Introduction

Risk decomposition methods from literature

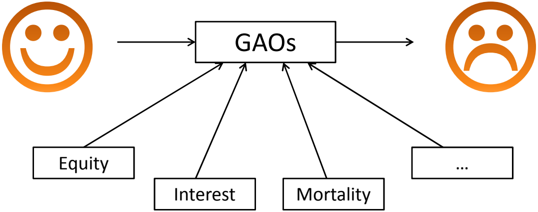
Martingale representation approach

Application to annuity conversion options

Outlook

Motivation

British insurance companies during the 1980s vs. 1990s:



Question: Which are the most relevant risk drivers?

Why is that important?

To be able to take adequate risk management strategies such as

- Product modifications
- Hedging

Research objectives

Situation:

- It is common to model the total risk by advanced stochastic models
- It is rarely discussed how to determine the most relevant risk driver

Our paper

(1) How to **allocate the randomness** of liabilities to different risk sources?

(2) How to **quantify** and **compare** the individual risk contributions?

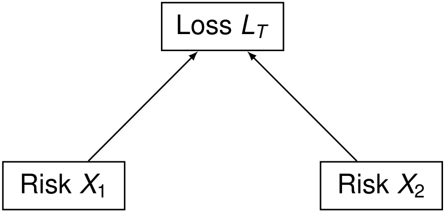
(3) What is the **dominating risk** in annuity conversion options?

In particular, we want to decompose the distribution under $\mathbb{P}$.

Setting

In the following:

- Insurance product with maturity T
- Time-T loss for the insurer: $\boldsymbol{L_T}$
- Two risk drivers: $\boldsymbol{X_1} := (X_1(t))_{t\geq 0}$ and $\boldsymbol{X_2} := (X_2(t))_{t\geq 0}$



Question: How to decompose L_T with respect to X_1 and X_2?

Variance decomposition approach

Step 1: $L_T = \underbrace{\mathrm{E}(L_T | X_1)}_{=R_1} + \underbrace{[L_T - \mathrm{E}(L_T | X_1)]}_{:=R_2}$

- R_1 represents the randomness of L_T caused by X_1
- R_2 represents the randomness of L_T caused by X_2

Step 2: $\mathrm{Var}(L_T) = \mathrm{Var}(R_1) + \mathrm{Var}(R_2)$

Desirable property: whole distribution of R_1 and R_2

- Bühlmann (1995): annual loss = financial loss + technical loss
- Example: $L_T = X_1(T)X_2(T)$, X_1, X_2 independent Brownian motions
  - $L_T = E(L_T|X_1) + [L_T - E(L_T|X_1)] = 0 + X_1(T)X_2(T)$
  - $L_T = E(L_T|X_2) + [L_T - E(L_T|X_2)] = 0 + X_1(T)X_2(T)$

Desirable property: symmetric definition

Modification

Idea: Use conditional expectations for a symmetric definition

Conditional expectation approach

The risk caused by risk driver X_i is measured by

$$R_i := E(L_T|X_i), \quad i = 1, \ldots, n.$$

- Example: $L_T = X_1(T)X_2(T)$, X_1, X_2 independent Brownian motions
  - $R_1 = E(L_T|X_1) = 0$
  - $R_2 = E(L_T|X_2) = 0$
- In general: $L_T \neq R_1 + \ldots + R_n$

Desirable property: $L_T - E(L_T) = R_1 + \ldots + R_n$

Further approaches

Sensitivity analysis

- Analyzing the effect of changes in the input parameters/variables on the insurer's loss
- Usually based on derivatives

Desirable property: Comparability between different risk drivers

Taylor expansion approach

- Function of random variables $\approx$ first-order Taylor expansion
- Christiansen (2007) extends this approach to an infinite setting
- Local method: expansion point is relevant

Desirable property: No problem-specific choices

Martingale representation approach (1)

In the following:

- $W = (W_1(t), \ldots, W_d(t))_{0 \le t \le T^*}$ d-dimensional Brownian motion
- $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le T^*}$ augmented natural filtration generated by W

Martingale representation theorem

If $M = (M(t))_{0 \le t \le T^*}$ is a martingale with respect to $\mathbb{G}$, then there exist unique $\mathbb{G}$-adapted processes $\Gamma_1, \ldots, \Gamma_d$ such that

$$M(t) = M(0) + \sum_{i=1}^{d} \int_0^t \Gamma_i(s) dW_i(s), \; 0 \le t \le T^*.$$

If L_T is $\mathcal{G}_T$-measurable and integrable, we define $M(t) := E\left(L_T \middle| \mathcal{G}_t \right)$ and obtain

$$L_T = \mathrm{E}\left(L_T\right) + \sum_{i=1}^{d} \underbrace{\int_0^T \Gamma_i(s) dW_i(s)}_{=: R_i}.$$

Martingale representation approach (2)

Special case: Itô's Lemma

Let $X = (X(t))_{0 \leq t \leq T^*}$ be an n-dimensional Itô process with dynamics

$$dX_i(t) = \mu_i(t, X(t))dt + \sum_{j=1}^{d} \sigma_{ij}(t, X(t))dW_j(t), \;\; i = 1, \dots, n.$$

If the martingale is of the very particular form $M(t) := E\left(L_T \middle| \mathcal{G}_t \right) = f(t, X(t))$, then Itô's lemma yields

$$L_T = \mathrm{E}\left(L_T\right) + \sum_{i=1}^{n} \underbrace{\int_0^T \frac{\partial f}{\partial x_i}(t, X(t)) \sum_{j=1}^{d} \sigma_{ij}(t, X(t))dW_j(t)}_{=:R_i}.$$

- **Sufficient conditions for $M(t) = f(t, X(t))$:**
  - $L_T = h(X(T))$ for some bounded, Borel-measurable function $h : \mathbb{R}^n \to \mathbb{R}$
  - $\mu_i(t, x)$ and $\sigma_{ij}(t, x)$ are Lipschitz $\Rightarrow$ X is a Markov process w.r.t. $\mathbb{G}$

$$\overset{\text{Factorization lemma}}{\Rightarrow} \quad M(t) := E\left(L_T \middle| \mathcal{G}_t \right) = E\left(h(X(T)) \middle| X(t) \right) = f(t, X(t))$$

Martingale representation approach (3)

Special case: Itô's Lemma

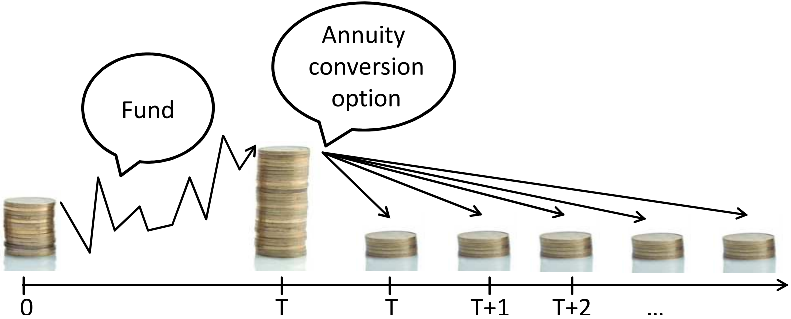
$$L_T = \mathrm{E}(L_T) + \sum_{i=1}^{n} \underbrace{\int_0^T \frac{\partial f}{\partial x_i}(t, X(t)) \sum_{j=1}^{d} \sigma_{ij}(t, X(t)) dW_j(t)}_{=:R_i}.$$

List of desirable properties:

- ✓ Whole distribution for each risk R_i
- ✓ Symmetric definition
- ✓ It holds: $L_T - E(L_T) = R_1 + \ldots + R_n$
- ✓ Comparability between different risk drivers
- ✓ No problem-specific choices

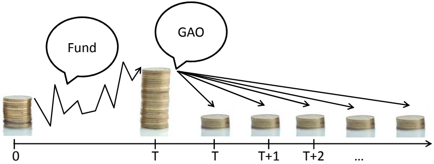
Annuity conversion options

- Additional feature of unit-linked deferred annuity contracts
- Guarantee the policyholder
  - at the beginning of the contract (at time 0)
  - certain **minimum conditions**
  - for **converting** the accumulated money into an annuity (at time *T*)



Guaranteed annuity option (GAO) (1)

- Caused serious solvency problems in the UK during the 1990s
- Special type of annuity conversion option:



Guaranteed annual annuity $=$ ***g* (conversion rate)** $\times$ A_T (account value)

Option payoff at time T (= insurer's loss)

$$L_T^{\text{GAO}} = \mathbb{1}_{\{\tau_x > T\}} \max\{gA_T a_T - A_T, 0\}$$
$$= \mathbb{1}_{\{\tau_x > T\}} gA_T \max\left\{a_T - \frac{1}{g}, 0\right\}$$

- τ_x: remaining lifetime of a policyholder aged x at time 0
- a_T: time-T value of an immediate annuity of unit amount per year

Guaranteed annuity option (GAO) (2)

Option payoff at time T (= insurer's loss)

$$L_T^{\text{GAO}} = \mathbb{1}_{\{\tau_x > T\}} g A_T \max\left\{a_T - \frac{1}{g}, 0\right\}$$

Risk	Implied by	Model
Fund risk	A_T	**GBM**
Interest risk	a_T	**Vasicek model**
Systematic mortality risk	a_T	Prudent mortality table
Unsystematic mortality risk	$\mathbb{1}_{\{\tau_x > T\}}$	∞-large portfolio

Martingale representation approach for GAOs (1)

Let W_S and W_r be two independent $\mathbb{P}$-Brownian motions

- **Fund value**: Geometric Brownian motion
$$dS(t) = \mu_S S(t)dt + \sigma_S S(t)dW_S(t), \;\; S(0) > 0$$
- **Short rate**: Vasicek model
$$dr(t) = \kappa(\theta - r(t))dt + \sigma_r dW_r(t), \;\; r(0) > 0$$

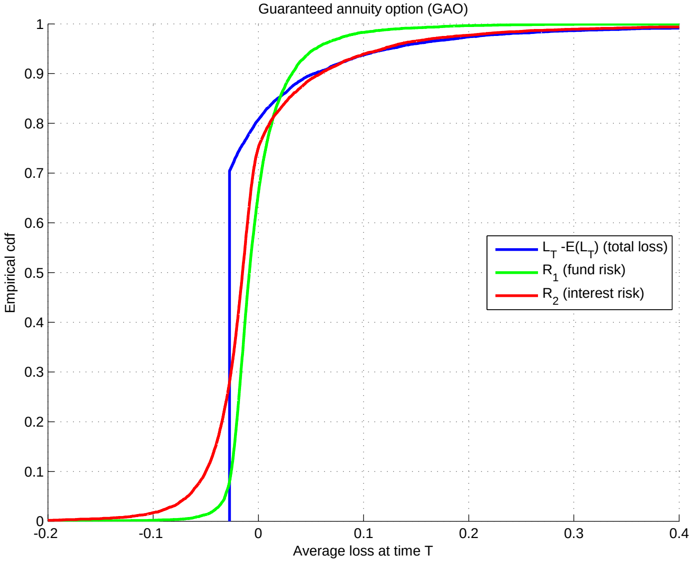
Martingale representation approach

With $f(t, x_1, x_2) := E(L_T^{\mathrm{GAO}} | S(t) = x_1, r(t) = x_2)$ and Itô's Lemma we obtain

$$L_T^{\mathrm{GAO}} - \mathrm{E}\left(L_T^{\mathrm{GAO}}\right) = \underbrace{\int_0^T \frac{\partial f}{\partial x_1}(t, S(t), r(t))\sigma_S S(t)dW_S(t)}_{=:R_1^{\mathrm{GAO}}}$$

$$+ \underbrace{\int_0^T \frac{\partial f}{\partial x_2}(t, S(t), r(t))\sigma_r dW_r(t)}_{=:R_2^{\mathrm{GAO}}}.$$

Martingale representation approach for GAOs (2)



Quantifying the risk contributions

Our paper

(1) How to allocate the randomness of liabilities to different risk sources?

(3) How to **quantify** and **compare** the individual risk contributions?

(3) What is the dominating risk in annuity conversion options?

Tail-Value-at-Risk (TVaR)

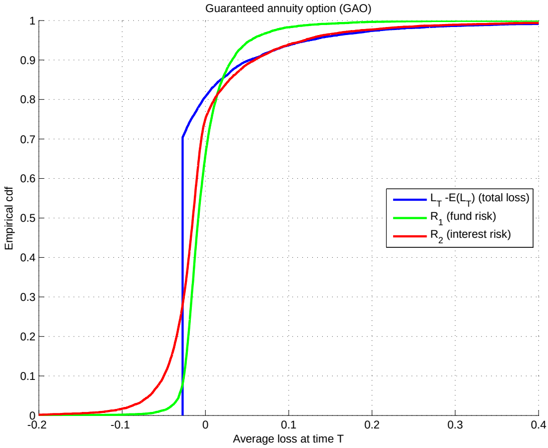
$$\mathrm{TVaR}_\alpha(X) = \mathrm{E}\left(X \mid X \geq \mathrm{Var}_\alpha(X)\right), \; 0 < \alpha < 1$$

TVaR approach

The risk contribution of X_i to the total risk $X = \sum_{i=1}^{n} X_i$ is quantified as

$$\mathrm{TVaR}_\alpha(X_i; X) := \mathrm{E}\left(X_i \mid X \geq \mathrm{Var}_\alpha(X)\right), \quad i = 1, \ldots, n \quad (\alpha \text{ fixed})$$

Martingale representation approach for GAOs (2)



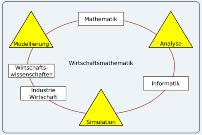
TVaR approach:
($\alpha = 0.99$)

$L_T^{\mathrm{GAO}} - \mathrm{E}(L_T^{\mathrm{GAO}})$	R_1^{GAO} (fund risk)	R_2^{GAO} (interest risk)
0.6440	0.1016	0.5407

Future research

- Stochastic mortality
  - Systematic mortality risk
  - Unsystematic mortality risk
- Application to other annuity conversion options
  - Modified GAOs
  - GMIBs
- Quantifying the individual risk contributions

Contact



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Thank you very much for your attention!

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