Decomposition of life insurance liabilities into risk factors – theory and application to annuity conversion options

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Research Training Group 1100
Introduction

Risk decomposition methods from literature

Martingale representation approach

Application to annuity conversion options

Outlook
Motivation

British insurance companies during the 1980s vs. 1990s:

Question: Which are the most relevant risk drivers?

Why is that important?

To be able to take adequate risk management strategies such as
  ▶ Product modifications
  ▶ Hedging
Research objectives

Situation:
- It is common to model the total risk by advanced stochastic models
- It is rarely discussed how to determine the most relevant risk driver

Our paper

1. How to allocate the randomness of liabilities to different risk sources?
2. How to quantify and compare the individual risk contributions?
3. What is the dominating risk in annuity conversion options?

In particular, we want to decompose the distribution under $\mathbb{P}$. 
Setting

In the following:

- Insurance product with maturity $T$
- Time-$T$ loss for the insurer: $L_T$
- Two risk drivers: $X_1 := (X_1(t))_{t \geq 0}$ and $X_2 := (X_2(t))_{t \geq 0}$

**Question:** How to decompose $L_T$ with respect to $X_1$ and $X_2$?
Variance decomposition approach

Step 1: \[ L_T = \underbrace{E(L_T|X_1)}_{=R_1} + \underbrace{[L_T - E(L_T|X_1)]}_{:=R_2} \]

- \( R_1 \) represents the randomness of \( L_T \) caused by \( X_1 \)
- \( R_2 \) represents the randomness of \( L_T \) caused by \( X_2 \)

Step 2: \[ \text{Var}(L_T) = \text{Var}(R_1) + \text{Var}(R_2) \]

Desirable property: whole distribution of \( R_1 \) and \( R_2 \)

- Bühlmann (1995): annual loss = financial loss + technical loss
- Example: \( L_T = X_1(T)X_2(T) \), \( X_1 \), \( X_2 \) independent Brownian motions
  - \( L_T = E(L_T|X_1) + [L_T - E(L_T|X_1)] = 0 + X_1(T)X_2(T) \)
  - \( L_T = E(L_T|X_2) + [L_T - E(L_T|X_2)] = 0 + X_1(T)X_2(T) \)

Desirable property: symmetric definition
Modification

Idea: Use conditional expectations for a symmetric definition

Conditional expectation approach

The risk caused by risk driver $X_i$ is measured by

$$R_i := E(L_T|X_i), \quad i = 1, \ldots, n.$$  

Example: $L_T = X_1(T)X_2(T)$, $X_1$, $X_2$ independent Brownian motions

- $R_1 = E(L_T|X_1) = 0$
- $R_2 = E(L_T|X_2) = 0$

In general: $L_T \neq R_1 + \ldots + R_n$

**Desirable property:** $L_T - E(L_T) = R_1 + \ldots + R_n$
Further approaches

Sensitivity analysis

- Analyzing the effect of changes in the input parameters/variables on the insurer’s loss
- Usually based on derivatives

Desirable property: Comparability between different risk drivers

Taylor expansion approach

- Function of random variables $\approx$ first-order Taylor expansion
- Christiansen (2007) extends this approach to an infinite setting
- Local method: expansion point is relevant

Desirable property: No problem-specific choices
Martingale representation approach (1)

In the following:

- \( \mathbf{W} = (W_1(t), \ldots, W_d(t))_{0 \leq t \leq T^*} \) \( d \)-dimensional Brownian motion
- \( \mathcal{G} = (\mathcal{G}_t)_{0 \leq t \leq T^*} \) augmented natural filtration generated by \( \mathbf{W} \)

Martingale representation theorem

If \( M = (M(t))_{0 \leq t \leq T^*} \) is a martingale with respect to \( \mathcal{G} \), then there exist unique \( \mathcal{G} \)-adapted processes \( \Gamma_1, \ldots, \Gamma_d \) such that

\[
M(t) = M(0) + \sum_{i=1}^{d} \int_{0}^{t} \Gamma_i(s) dW_i(s), \ 0 \leq t \leq T^*.
\]

If \( L_T \) is \( \mathcal{G}_T \)-measurable and integrable, we define \( M(t) := E(L_T | \mathcal{G}_t) \) and obtain

\[
L_T = E(L_T) + \sum_{i=1}^{d} \int_{0}^{T} \Gamma_i(s) dW_i(s) =: R_i
\]
Martingale representation approach (2)

Special case: Itô’s Lemma

Let $X = (X(t))_{0 \leq t \leq T^*}$ be an $n$-dimensional Itô process with dynamics

$$dX_i(t) = \mu_i(t, X(t))dt + \sum_{j=1}^{d} \sigma_{ij}(t, X(t))dW_j(t), \ i = 1, \ldots, n.$$ 

If the martingale is of the very particular form $M(t) := E(L_T|G_t) = f(t, X(t))$, then Itô’s lemma yields

$$L_T = E(L_T) + \sum_{i=1}^{n} \int_{0}^{T} \frac{\partial f}{\partial X_i}(t, X(t)) \sum_{j=1}^{d} \sigma_{ij}(t, X(t))dW_j(t).$$

Sufficient conditions for $M(t) = f(t, X(t))$: 

- $L_T = h(X(T))$ for some bounded, Borel-measurable function $h: \mathbb{R}^n \to \mathbb{R}$
- $\mu_i(t, x)$ and $\sigma_{ij}(t, x)$ are Lipschitz $\Rightarrow$ $X$ is a Markov process w.r.t. $G$

Factorization lemma

$$M(t) := E(L_T|G_t) = E(h(X(T))|X(t)) = f(t, X(t))$$
Martingale representation approach (3)

Special case: Itô’s Lemma

\[
L_T = E(L_T) + \sum_{i=1}^{n} \int_{0}^{T} \frac{\partial f}{\partial x_i}(t, X(t)) \sum_{j=1}^{d} \sigma_{ij}(t, X(t)) dW_j(t) =: R_i
\]

List of desirable properties:

✓ Whole distribution for each risk \( R_i \)
✓ Symmetric definition
✓ It holds: \( L_T - E(L_T) = R_1 + \ldots + R_n \)
✓ Comparability between different risk drivers
✓ No problem-specific choices
Annuity conversion options

- Additional feature of unit-linked deferred annuity contracts
- Guarantee the policyholder
  - at the beginning of the contract (at time 0)
  - certain **minimum conditions**
  - for **converting** the accumulated money into an annuity (at time $T$)
Guaranteed annuity option (GAO) (1)

- Caused serious solvency problems in the UK during the 1990s
- Special type of annuity conversion option:

\[ L_T^{GAO} = \mathbb{1}_{\{\tau_x > T\}} \max \{ gA_T a_T - A_T, 0 \} \]

\[ = \mathbb{1}_{\{\tau_x > T\}} gA_T \max \left\{ a_T - \frac{1}{g}, 0 \right\} \]

- \( \tau_x \): remaining lifetime of a policyholder aged \( x \) at time 0
- \( a_T \): time-\( T \) value of an immediate annuity of unit amount per year

Guaranteed annual annuity = \( g \) (conversion rate) \( \times \) \( A_T \) (account value)
Guaranteed annuity option (GAO) (2)

Option payoff at time $T$ (= insurer’s loss)

$$L_T^{GAO} = \mathbb{1}_{\{\tau_x > T\}} gA_T \max \left\{ a_T - \frac{1}{g}, 0 \right\}$$

<table>
<thead>
<tr>
<th>Risk</th>
<th>Implied by</th>
<th>Model</th>
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</thead>
<tbody>
<tr>
<td><strong>Fund risk</strong></td>
<td>$A_T$</td>
<td><strong>GBM</strong></td>
</tr>
<tr>
<td><strong>Interest risk</strong></td>
<td>$a_T$</td>
<td><strong>Vasicek model</strong></td>
</tr>
<tr>
<td>Systematic mortality risk</td>
<td>$a_T$</td>
<td><strong>Prudent mortality table</strong></td>
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<tr>
<td>Unsystematic mortality risk</td>
<td>$\mathbb{1}_{{\tau_x &gt; T}}$</td>
<td>$\infty$-large portfolio</td>
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Martingale representation approach for GAOs (1)

Let $W_S$ and $W_r$ be two independent $\mathbb{P}$-Brownian motions

- **Fund value**: Geometric Brownian motion
  \[ dS(t) = \mu_S S(t)dt + \sigma_S S(t)dW_S(t), \quad S(0) > 0 \]

- **Short rate**: Vasicek model
  \[ dr(t) = \kappa(\theta - r(t))dt + \sigma_r dW_r(t), \quad r(0) > 0 \]

Martingale representation approach

With $f(t, x_1, x_2) := E(L_T^{\text{GAO}}|S(t) = x_1, r(t) = x_2)$ and Itô’s Lemma we obtain

\[
L_T^{\text{GAO}} - E(L_T^{\text{GAO}}) = \int_0^T \frac{\partial f}{\partial x_1}(t, S(t), r(t))\sigma_S S(t)dW_S(t) =: R_1^{\text{GAO}} \\
+ \int_0^T \frac{\partial f}{\partial x_2}(t, S(t), r(t))\sigma_r dW_r(t) =: R_2^{\text{GAO}}
\]
Martingale representation approach for GAOs (2)
Quantifying the risk contributions

Our paper

1. How to allocate the randomness of liabilities to different risk sources?
2. How to **quantify** and **compare** the individual risk contributions?
3. What is the dominating risk in annuity conversion options?

Tail-Value-at-Risk (TVaR)

\[
\text{TVaR}_\alpha(X) = \mathbb{E}(X \mid X \geq \text{Var}_\alpha(X)), \quad 0 < \alpha < 1
\]

TVaR approach

The risk contribution of \( X_i \) to the total risk \( X = \sum_{i=1}^{n} X_i \) is quantified as

\[
\text{TVaR}_\alpha(X_i; X) := \mathbb{E}(X_i \mid X \geq \text{Var}_\alpha(X)), \quad i = 1, \ldots, n \quad (\alpha \text{ fixed})
\]
Martingale representation approach for GAOs (2)

TVaR approach: \( L_{T}^{GAO} - E(L_{T}^{GAO}) \)  
\( R_{1}^{GAO} \) (fund risk)  \( R_{2}^{GAO} \) (interest risk)  
\( \alpha = 0.99 \)  
0.6440  0.1016  0.5407
Future research

- Stochastic mortality
  - Systematic mortality risk
  - Unsystematic mortality risk

- Application to other annuity conversion options
  - Modified GAOs
  - GMIBs

- Quantifying the individual risk contributions
Thank you very much for your attention!
Literature


