Multi-year non-life insurance risk of dependent lines of business
The multivariate additive loss reserving model

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Agenda

Multi-year non-life insurance risk for dependent portfolios

The multivariate additive loss reserving model

Analytical risk estimators

Case study

Conclusion
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Risk horizon

- **Risk horizon** for non-life insurance risk

- Traditional **ultimo view** measures uncertainty stemming from all future occurrences and payments until final settlement at $T = T^*$
Risk horizon

- **Risk horizon** for non-life insurance risk

- Modern regulation requires a one-year view, i.e. uncertainty in solely next year’s claims settlement and in updating reserves for outstanding payments
Risk horizon

- **Risk horizon** for non-life insurance risk

- A general multi-year view builds a bridge between both horizons: uncertainty in settling $m = 1, \ldots, T^* - n$ years and in updating remaining reserves
  - Forward-looking risk management
  - Projection of SCR as part of ORSA and for risk margin calculation
Single portfolio

- Uncertainty in claims development result $\hat{CDR}^{(n\to n+m)}$ of a single portfolio
  - **Analytical approaches**: Closed-form estimators for the mean squared error of prediction (msep) of $\hat{CDR}^{(n\to n+m)}$ in distribution-free stochastic reserving models
    - additive loss reserving model (Diers and Linde, 2013)
    - Mack chain ladder (CL) model (Diers et al., 2016)
  - **Bootstrap methods**: Stochastic $m$-year re-reserving (“actuary in a box”) in Diers et al. (2013) to estimate a full predictive distribution of $\hat{CDR}^{(n\to n+m)}$
    - generalizing the one-year bootstrap by Ohlsson and Lauzeningks (2009)
Dependent portfolios

- Dependencies in claims development among portfolios
- Dependencies in CDR among portfolios
- Aggregated CDR

**Goal**
- Estimate joint risk subject to dependencies among CDR
Dependent portfolios

**Two-step approach**
- Analyze risk in each portfolio’s CDR individually
- Analyze dependencies among individual CDR in a subsequent step
Dependent portfolios

Single-step approach

- Analyze risk in aggregated CDR subject to dependencies among claims developments of individual portfolios (i.e. no aggregated portfolio!)
Agenda

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Multivariate extended additive loss reserving model

**Definition**

The **multivariate extended additive loss reserving model** is defined through

- $s_{i,k}$, $i = 1, \ldots, n + m$, $k = 1, \ldots, n$ are independent
- For each $k = 1, \ldots, n$ there exists $\mu_k = \left(\mu_k^{(1)}, \ldots, \mu_k^{(q)}\right)'$ such that $\mathbb{E}(s_{i,k}) = V_i \mu_k$.
- For each $k = 1, \ldots, n$ there exists a symmetric positive definite $q \times q$ matrix $\Sigma_k$ such that $\nabla(s_{i,k}) = V_i^{1/2} \Sigma_k V_i^{1/2}$.

- Features compared to Hess et al. (2006)
  - We assume global independence
  - We integrate future accident years
Multi-year framework

Definition

Given $\Delta_n \rightarrow \Delta_{n+m}$, the $m$-year (observable) \textbf{claims development result} for a single accident year $i \in \{1, \ldots, n+m\}$ is

$$\widehat{\text{CDR}}_i(n\rightarrow n+m) = (n)\hat{u}_i - (n+m)\hat{u}_i.$$  

- In the multivariate extended additive model,

$$\widehat{\text{CDR}}_i(n\rightarrow n+m) = V_i \sum_{k=\max\{1, n-i+2\}}^{\min\{n+m-i+1, n\}} \left((n)\hat{\mu}_k - m_{i,k}\right)$$

$$+ V_i \sum_{k=n+m-i+2}^{n} \left((n)\hat{\mu}_k - (n+m)\hat{\mu}_k\right)$$

- Risk measure: msep
Agenda

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Conclusion
Estimator for risk in a single accident year

**Theorem**

The unconditional covariance matrix of the $m$-year CDR for one accident year $i \in \{2, \ldots, n + m\}$ is given by

$$
\text{V} \left( \widehat{\text{CDR}}_{i}^{(n \to n+m)} \right) = V_i \left[ \min\{n+m-i+1,n\} \sum_{k=\max\{1,n-i+2\}}^{n} \left( \begin{pmatrix} n \end{pmatrix} V \Sigma_{\leq k}^{-1} + V_i^{-1/2} \Sigma_k V_i^{-1/2} \right) 
+ \sum_{k=n+m-i+2}^{n} \left( \begin{pmatrix} n \end{pmatrix} V \Sigma_{\leq k}^{-1} - \begin{pmatrix} n+m \end{pmatrix} V \Sigma_{\leq k}^{-1} \right) \right] V_i
$$

- If $\Sigma_k$ is unknown, substitute by suitable estimator $\widehat{\Sigma}_k$
Estimator for non-life insurance risk

- More generally, we obtain estimators for
  - non-life insurance risk: \( \mathbb{V}\left( \hat{CDR}^{(n\rightarrow n+m)} \right) \)
  - reserve risk: \( \mathbb{V}\left( \hat{CDR}_{PY}^{(n\rightarrow n+m)} \right) \)
  - premium risk: \( \mathbb{V}\left( \hat{CDR}_{NY}^{(n\rightarrow n+m)} \right) \)
- Special cases for
  - one-year view: \( m = 1 \)
  - ultimo view: \( m = i - 1 \) (single accident year \( i \)) or \( m = n - 1 \) (reserve risk)
- Aggregated risk
  - Use, e.g., \( \mathbb{V}\left( \hat{aCDR}_i^{(n\rightarrow n+m)} \right) = 1' \mathbb{V}\left( \hat{CDR}_i^{(n\rightarrow n+m)} \right) 1 \)
  - For aggregated reserve risk in ultimo view, we obtain the same result as in Merz and Wüthrich (2009)
- Single portfolio
  - Use \( q = 1 \)
Estimator for non-life insurance risk

- Effective correlation between reserve and premium risk
  - Estimator for aggregated business
    \[
    \text{Corr}\left(\hat{\text{aCDR}}_{\text{PY}}^{(n\rightarrow n+m)}, \hat{\text{aCDR}}_{\text{NY}}^{(n\rightarrow n+m)}\right) = \frac{\text{V}\left(\hat{\text{aCDR}}^{(n\rightarrow n+m)}\right) - \text{V}\left(\hat{\text{aCDR}}_{\text{PY}}^{(n\rightarrow n+m)}\right) - \text{V}\left(\hat{\text{aCDR}}_{\text{NY}}^{(n\rightarrow n+m)}\right)}{2\sqrt{\text{V}\left(\hat{\text{aCDR}}_{\text{PY}}^{(n\rightarrow n+m)}\right) \text{V}\left(\hat{\text{aCDR}}_{\text{NY}}^{(n\rightarrow n+m)}\right)}}
    \]
  - Analogous estimator for marginal effective correlation
  - Similar logic for effective correlation of fixed risk component between marginal portfolios
- Risk in future one-year view \(\text{V}\left(\hat{\text{CDR}}^{(n+t\rightarrow n+t+1)}\right)\)
  - Closed-form estimators available
    \[
    \text{V}\left(\hat{\text{CDR}}^{(n\rightarrow n+t+1)}\right) = \text{V}\left(\hat{\text{CDR}}^{(n\rightarrow n+t)}\right) + \text{V}\left(\hat{\text{CDR}}^{(n+t\rightarrow n+t+1)}\right)
    \]
Agenda

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Data

- **Data** used in Braun (2004), Merz and Wüthrich (2009), Ludwig and Schmidt (2010), Diers and Linde (2013)
  - $q = 2$ portfolios
    - general third-party liability
    - motor vehicle third-party liability
  - $n = 14$ historic accident years with volumes
  - Include volumes for $m = 5$ future accident years
- **Gauss-Markov parameter estimation** as in Merz and Wüthrich (2009)
Selected results

- Parameter estimates

Parameter estimates in the multivariate additive loss reserving model

Notation: $\hat{\sigma}_k^{(p)}$ denote estimated standard errors at $k$ for $p = 1, 2$, i.e. $\hat{\sigma}_k^{(p)} := \sqrt{\hat{\Sigma}_k^{(p,p)}}$, and $\hat{\rho}_k^{(1,2)} := \frac{\hat{\Sigma}_k^{(1,2)}}{\hat{\sigma}_k^{(1)} \hat{\sigma}_k^{(2)}}$ the estimated correlation coefficient.

Estimates are calculated as in Merz and Wüthrich (2009, Table 4). For our extended model, we additionally estimate parameters for $k = 1$.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Development years $k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\mu}_k^{(1)}$</td>
<td>0.08860 0.19974 0.20640 0.17493 0.12119 0.08452 0.02476 0.01441 0.01195 0.00614 0.00428 0.00529 0.00371</td>
</tr>
<tr>
<td>$\hat{\mu}_k^{(2)}$</td>
<td>0.26771 0.32899 0.16172 0.09061 0.05572 0.03170 0.01550 0.00910 0.00017 0.00354 −0.0051 0.00354 −0.00097 −0.00026</td>
</tr>
<tr>
<td>$\hat{\sigma}_k^{(1)}$</td>
<td>22.17661 31.57523 20.03153 14.42171 18.92315 13.63922 13.90884 5.78725 7.16307 12.20596 6.09168 1.83922 0.55530 0.16766</td>
</tr>
<tr>
<td>$\hat{\sigma}_k^{(2)}$</td>
<td>27.95800 27.73945 18.19876 15.17124 15.99662 11.73743 5.17320 4.69929 2.05411 4.96015 1.34770 2.99827 1.34770 0.60578</td>
</tr>
<tr>
<td>$\hat{\rho}_k^{(1,2)}$</td>
<td>0.35146 −0.02654 0.84893 0.59216 0.37111 0.34034 0.31262 −0.10467 0.75529 0.33235 0.66612 −0.13921 0.14399 0.14894</td>
</tr>
</tbody>
</table>
**Selected results**

- **Best estimate reserves**
  - No material difference between univariate and multivariate models

### Best estimate reserves for outstanding and future claims by line of business and aggregated for overall business.

Best estimate reserves are calculated via the multivariate additive loss reserving model ("multivariate") and the univariate additive loss reserving model applied individually to single triangles ("univariate (marginal)") and to the overall triangle when first aggregating both triangles by adding all entries ("univariate (overall)").

<table>
<thead>
<tr>
<th>Reserve component</th>
<th>General liability business</th>
<th>Auto liability business</th>
<th>Overall business</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Multivariate</td>
<td>Univariate (individual)</td>
<td>Multivariate</td>
</tr>
<tr>
<td>Prior years PY = {1, ..., 14}</td>
<td>6,315,254</td>
<td>6,311,503</td>
<td>2,050,866</td>
</tr>
<tr>
<td>Future years NY = {15, ..., 19} of which $i = n + 1 = 15$</td>
<td>14,594,119</td>
<td>14,591,110</td>
<td>8,311,027</td>
</tr>
<tr>
<td>All years PY $\cup$ NY = {1, ..., 19}</td>
<td>20,909,372</td>
<td>20,902,613</td>
<td>10,361,893</td>
</tr>
<tr>
<td>of which PY $\cup$ {15}</td>
<td>8,916,977</td>
<td>8,912,690</td>
<td>3,563,255</td>
</tr>
</tbody>
</table>
Selected results

- **Reserve risk**
  - Considerably higher in multivariate model
  - Stable correlation

---

### Estimated reserve risk by line of business and aggregated for overall business.

Reserve risk is estimated via prediction errors (PE, standard errors of prediction variance) for all prior years and coefficients of variation (CoV, in terms of best estimate reserves) by models as in previous table. For the multivariate model we also calculate the effective linear correlation between both portfolios (Corr).

<table>
<thead>
<tr>
<th>Multi-year view</th>
<th>General liability business</th>
<th>Auto liability business</th>
<th>Overall business</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Multivariate (individual)</td>
<td>Multivariate (individual)</td>
<td>Multivariate (individual)</td>
</tr>
<tr>
<td>$m$</td>
<td>PE CoV</td>
<td>PE CoV</td>
<td>PE CoV</td>
</tr>
<tr>
<td>1</td>
<td>126,647 2.0% 126,707 2.0%</td>
<td>68,738 3.4% 68,788 3.4%</td>
<td>161,656 1.9% 30.8%</td>
</tr>
<tr>
<td>2</td>
<td>161,783 2.6% 161,846 2.6%</td>
<td>86,051 4.2% 86,103 4.2%</td>
<td>207,972 2.5% 34.7%</td>
</tr>
<tr>
<td>3</td>
<td>182,838 2.9% 182,897 2.9%</td>
<td>95,830 4.7% 95,880 4.7%</td>
<td>233,761 2.8% 34.3%</td>
</tr>
<tr>
<td>4</td>
<td>196,249 3.1% 196,309 3.1%</td>
<td>101,398 4.9% 101,446 5.0%</td>
<td>249,429 3.0% 33.7%</td>
</tr>
<tr>
<td>5</td>
<td>204,234 3.2% 204,294 3.2%</td>
<td>103,953 5.1% 104,000 5.1%</td>
<td>258,155 3.1% 33.3%</td>
</tr>
<tr>
<td>6</td>
<td>209,462 3.3% 209,525 3.3%</td>
<td>105,057 5.1% 105,104 5.1%</td>
<td>263,461 3.1% 32.9%</td>
</tr>
<tr>
<td>7</td>
<td>212,578 3.4% 212,643 3.4%</td>
<td>105,793 5.2% 105,840 5.2%</td>
<td>266,686 3.2% 32.8%</td>
</tr>
<tr>
<td>8</td>
<td>214,854 3.4% 214,919 3.4%</td>
<td>106,285 5.2% 106,332 5.2%</td>
<td>269,088 3.2% 32.7%</td>
</tr>
<tr>
<td>9</td>
<td>216,240 3.4% 216,305 3.4%</td>
<td>106,638 5.2% 106,685 5.2%</td>
<td>270,555 3.2% 32.7%</td>
</tr>
<tr>
<td>10</td>
<td>216,520 3.4% 216,585 3.4%</td>
<td>106,781 5.2% 106,828 5.2%</td>
<td>270,873 3.2% 32.6%</td>
</tr>
<tr>
<td>11</td>
<td>216,546 3.4% 216,611 3.4%</td>
<td>106,878 5.2% 106,925 5.2%</td>
<td>270,927 3.2% 32.6%</td>
</tr>
<tr>
<td>12</td>
<td>216,548 3.4% 216,613 3.4%</td>
<td>106,897 5.2% 106,944 5.2%</td>
<td>270,937 3.2% 32.6%</td>
</tr>
<tr>
<td>13</td>
<td>216,548 3.4% 216,613 3.4%</td>
<td>106,900 5.2% 106,947 5.2%</td>
<td>270,939 3.2% 32.6%</td>
</tr>
</tbody>
</table>
Selected results

- **Premium risk** assuming one future year
  - Considerably higher in multivariate model

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**Estimated premium risk of one future year by line of business and aggregated for overall business.**

Premium risk is estimated via prediction errors (PE, standard errors of prediction variance) for the next future year and coefficients of variation (CoV, in terms of best estimate reserves) by models as in previous table.

For the multivariate model we also calculate the effective linear correlation between both portfolios (Corr).

<table>
<thead>
<tr>
<th>Multi-year view</th>
<th>General liability business</th>
<th>Auto liability business</th>
<th>Overall business</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Multivariate</td>
<td>Univariate (individual)</td>
<td>Multivariate</td>
</tr>
<tr>
<td></td>
<td>PE</td>
<td>CoV</td>
<td>PE</td>
</tr>
<tr>
<td><strong>m</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>42,704</td>
<td>1.6%</td>
<td>42,718</td>
</tr>
<tr>
<td>2</td>
<td>79,973</td>
<td>3.1%</td>
<td>80,000</td>
</tr>
<tr>
<td>3</td>
<td>84,440</td>
<td>3.2%</td>
<td>84,465</td>
</tr>
<tr>
<td>4</td>
<td>90,964</td>
<td>3.5%</td>
<td>90,987</td>
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<tr>
<td>5</td>
<td>94,232</td>
<td>3.6%</td>
<td>94,255</td>
</tr>
<tr>
<td>6</td>
<td>97,411</td>
<td>3.7%</td>
<td>97,434</td>
</tr>
<tr>
<td>7</td>
<td>98,033</td>
<td>3.8%</td>
<td>98,056</td>
</tr>
<tr>
<td>8</td>
<td>98,881</td>
<td>3.8%</td>
<td>98,902</td>
</tr>
<tr>
<td>9</td>
<td>101,130</td>
<td>3.9%</td>
<td>101,151</td>
</tr>
<tr>
<td>10</td>
<td>101,680</td>
<td>3.9%</td>
<td>101,701</td>
</tr>
<tr>
<td>11</td>
<td>101,730</td>
<td>3.9%</td>
<td>101,751</td>
</tr>
<tr>
<td>12</td>
<td>101,734</td>
<td>3.9%</td>
<td>101,756</td>
</tr>
<tr>
<td>13</td>
<td>101,735</td>
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<td>101,756</td>
</tr>
<tr>
<td>14</td>
<td></td>
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Selected results

- Further analyses results possible

<table>
<thead>
<tr>
<th>Multi-year view ( m )</th>
<th>General liability business</th>
<th>Auto liability business</th>
<th>Overall business</th>
<th>Process</th>
<th>Estimation</th>
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<td>17.0%</td>
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<td>70.7%</td>
<td>29.3%</td>
</tr>
<tr>
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<td>33.6%</td>
<td>22.5%</td>
<td>30.3%</td>
<td>61.4%</td>
<td>38.6%</td>
</tr>
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<td>30.2%</td>
<td>54.5%</td>
<td>45.5%</td>
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<td>29.2%</td>
<td>50.0%</td>
<td>50.0%</td>
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<td>53.2%</td>
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<td>43.0%</td>
<td>57.0%</td>
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<tr>
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<td>41.9%</td>
<td>58.1%</td>
</tr>
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<td>21.4%</td>
<td>28.3%</td>
<td>41.6%</td>
<td>58.4%</td>
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<td>11</td>
<td>31.3%</td>
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<td>28.2%</td>
<td>41.6%</td>
<td>58.4%</td>
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<td>41.5%</td>
<td>58.5%</td>
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<td>13</td>
<td>31.3%</td>
<td>21.4%</td>
<td>28.2%</td>
<td>41.5%</td>
<td>58.5%</td>
</tr>
<tr>
<td>14</td>
<td>31.3%</td>
<td>21.4%</td>
<td>28.2%</td>
<td>41.5%</td>
<td>58.5%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Multivariate</th>
<th>Proportion of process and estimation variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overall business</td>
<td>Reserve risk</td>
</tr>
<tr>
<td>Process</td>
<td>Estimation</td>
</tr>
<tr>
<td>86.9%</td>
<td>13.1%</td>
</tr>
</tbody>
</table>
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Conclusion
Summary

- We derived **closed-form risk estimators** in the multivariate additive loss reserving model that combine
  - evaluation of both reserve and premium risk and their dependencies for all portfolios in one approach,
  - quantification of all risk types for all marginal portfolios as well as the entire business including the risk loading stemming from portfolio dependencies,
  - detailed evolution of all risk components from one-year to ultimo view and extraction of one-year risk at future accounting dates, and
  - decomposition of risk into its sources of process and estimation uncertainty
Advantages and limitations

▶ **Advantages**
  ▶ Easily computable and interpretable
  ▶ Consistent with distribution-free reserving process
  ▶ Valuable for adequate strategic and regulatory risk modeling
    ▶ Forward looking view in Solvency II
    ▶ Comparative analysis of prescribed and undertaking-specific parameters (e.g. correlation between one-year reserve and premium risk in standard formula)

▶ **Limitations**
  ▶ No stand-alone risk evaluation
    ▶ No VaR/TVaR or predictive distribution
  ▶ Only valid if model fits the data
    ▶ Uncommon choice for marginal portfolios
    ▶ Restrictive independence assumption
  ▶ Challenging estimation of covariance matrices
Advantages and limitations

Future research

- Fruitful as starting point for more sophisticated methods
  - Combination with other reserving models
  - Calibration of distributions for CDR
  - Input parameters for simulation-based approaches (stochastic re-reserving)
Contact details

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Literature I


Literature II


Literature III
