Risk analysis of annuity conversion options with a special focus on decomposing risk

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- Berlin, June 2018
Risk analysis of annuity conversion options with a special focus on decomposing risk

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Introduction

Risk analysis of annuity conversion options

Risk decomposition methods from literature

MRT decomposition

Application of MRT decomposition to annuity conversion options

Contact details
About the speaker

- Master of Science (Mathematics, Illinois State University, USA, 2010)
- Diploma (Mathematics and Management, University of Ulm, 2011)
- Ph.D. (University of Ulm, 2017)
- since 2015 consultant at Allianz Pension Consult
  (Allianz Pension Consult is a subsidiary of Allianz Lebensversicherungs-AG, the largest life insurance company in Germany, with main focus on structuring and arranging tailor-made solutions in the field of occupational pensions for medium-sized and large companies)
- junior member of the German Society for Actuarial and Financial Mathematics (DGVFM)
- candidate for the membership of the German Society of Actuaries (DAV)
Institut für Finanz- und Aktuarwissenschaften (ifa)

- ifa is an independent actuarial consulting firm.
- Our consulting services in all lines of insurance business include:
  - typical actuarial tasks and actuarial modelling
  - insurance product development
  - risk management, Solvency II, asset liability management
  - data analytics
  - market entries (cross-border business, setup of new insurance companies, Fintechs)
  - professional education
  - academic research on actuarial topics of practical relevance
- located in Ulm, Germany
- currently about 30 consultants
- academic cooperation with the University of Ulm (offering the largest actuarial program in Germany)

Dr. Alexander Kling

- joined ifa in 2003
- qualified actuary (German Association of Actuaries DAV, 2007)
- Master of Science (University of Wisconsin, Milwaukee, 2002)
- Master of Science (Mathematics and Management, University of Ulm, 2003)
- Ph.D. (University of Ulm, 2007)
- lecturer at Ludwig-Maximilians-Universität München, University of Ulm, German Actuarial Academy (DAA), European Actuarial Academy (EAA)
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Annuity conversion options

(Unit-linked) deferred annuities

- Money is allocated in some fund during a deferment period.
- At the end of the deferment period, the accumulated fund value is converted into a lifelong annuity.

Different annuity conversion options

- Guaranteed annuity option (GAO)
- GAO on a limited amount (Limit)
- Guaranteed minimum income benefit (GMIB)

Annuity conversion options are influenced by various risk sources such as:
- equity, interest rate, and mortality

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Introduction
Annuity conversion options

Existing literature on annuity conversion options
- measure the total risk by advanced stochastic models
- typically no decomposition of the total risk into risk factors

Our research interests
- (1) Theory:
  - How can the randomness of liabilities be allocated to different risk sources?
- (2) Application to annuity conversion options:
  - What is the dominating risk in annuity conversion options?
  - What is the relative importance of different risk sources?
- (3) Risk management of annuity conversion options:
  - How can the single risks be managed by product design or internal hedging?

Our contributions
- Risk analysis of annuity conversion options in a stochastic mortality environment
  - Katja Schilling, Alexander Kling, Jochen Ruß (2014)
  - ASTIN Bulletin 44 (2), 197 - 236.
- Decomposing life insurance liabilities into risk factors
  - Katja Schilling, Daniel Bauer, Marcus C. Christiansen, Alexander Kling (2018)
  - under review: Management Science
- Comparing financial and biometric risks in annuity conversion options via the MRT decomposition
  - Katja Schilling (2018)
  - under review: Insurance: Mathematics and Economics
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Different annuity conversion options

Guaranteed annuity option (GAO)
- minimum conversion rate $g$ for converting the account value into a lifelong annuity at time $T$

$$L_{T}^{GAO,i} = \mathbb{1}_{(\tau_{x} > T)} \cdot g \cdot A_{T} \cdot \max\{a_{T} - \frac{1}{g}, 0\}$$

GAO with limit (Limit)
- upper bound $L$ (limit) to which the conversion rate $g$ at most applies

$$L_{T}^{Limit,i} = \mathbb{1}_{(\tau_{x} > T)} \cdot g \cdot \min\{A_{T}; L\} \cdot \max\{a_{T} - \frac{1}{g}, 0\}$$

Guaranteed minimum income benefit (GMIB)
- fixed minimum annuity amount $M(= g \cdot G)$

$$L_{T}^{GMIB,i} = \mathbb{1}_{(\tau_{x} > T)} \cdot \max\{g \cdot G \cdot a_{T} - A_{T}, 0\}$$

Notation
- $T$: deferment period/retirement date
- $x$: policyholder’s age at inception of the contract ($t = 0$)
- $\tau_{x}$: remaining lifetime
- $A_{T}$: account value at the end of the deferment period
- $a_{T}$: present value of an immediate annuity of amount 1 p.a.
Risk analysis of annuity conversion options
Insurer’s strategies and stochastic model

### Risk management strategies

<table>
<thead>
<tr>
<th>No hedging</th>
<th>Hedging</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>No option fee</strong></td>
<td>Strategy A</td>
</tr>
<tr>
<td><strong>Option fee</strong></td>
<td>Strategy B</td>
</tr>
</tbody>
</table>

- **Strategy A**
  - Insurer charges no option fee and does not hedge.

- **Strategy B**
  - Insurer charges an option fee which is simply invested in money market instruments (no hedging).

- **Strategy C**
  - Insurer charges an option fee to buy a static hedge against the financial risk during the deferment period.
  - Assumption: option fee = hedging costs

### Stochastic model

#### Fund
- **Geometric Brownian motion**
  \[ dS(t) = (r(t) + \lambda_s) \cdot S(t) \, dt + \sigma_s \cdot S(t) \, dW_S(t), \quad S(0) > 0 \]

#### Interest rate
- **Cox-Ingersoll-Ross model**
  \[ dr(t) = \kappa \cdot (\theta - r(t)) \, dt + \sigma_r \cdot \sqrt{r(t)} \, dW_r(t), \quad r(0) > 0 \]

#### Stochastic mortality
- 6-factor forward model (cf. Bauer et al., 2008)
  \[ d\mu(t,T,x) = \alpha(t,T,x) \, dt + \sigma(t,T,x) \, dW_\mu(t), \quad \mu(0,T,x) > 0 \]
Risk analysis of annuity conversion options
Sample results

Risk of different annuity conversion options **without hedging**
(under strategy A)

<table>
<thead>
<tr>
<th></th>
<th>GAO</th>
<th>Limit</th>
<th>GMIB</th>
</tr>
</thead>
<tbody>
<tr>
<td>risk ($TVaR_{0.99}$)</td>
<td>0.7362</td>
<td>0.2214</td>
<td>0.7739</td>
</tr>
<tr>
<td>option value</td>
<td>0.0167</td>
<td>0.0096</td>
<td>0.1361</td>
</tr>
</tbody>
</table>

- Loss probability for GMIB much higher than for the other annuity conversion options
- Risk ($TVaR_{0.99}$) similar for GMIB and GAO
  - Limit has a much lower risk
- Option value does not reflect the risk of the annuity conversion option

Cumulative distribution function of insurer’s loss

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Risk analysis of annuity conversion options with a special focus on decomposing risk
Risk analysis of annuity conversion options

Sample results

Risk of GMIB guarantee under different risk management strategies

<table>
<thead>
<tr>
<th></th>
<th>strategy A</th>
<th>strategy B</th>
<th>strategy C</th>
</tr>
</thead>
<tbody>
<tr>
<td>risk ($TVaR_{0.99}$)</td>
<td>0.7739</td>
<td>0.5962</td>
<td>0.2385</td>
</tr>
<tr>
<td>option value</td>
<td>0.1361</td>
<td>0.1708</td>
<td>0.1708</td>
</tr>
</tbody>
</table>

- Loss probability and risk can significantly be reduced by risk management (strategies B and C)
- Risk under strategy B (option fee but no hedging) still significant
- Significant risk reduction for strategy C (option fee and hedging)
- Option value is increased if a guarantee fee is charged
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Variance decomposition approach

**Definition** (given two sources of risk $X_1$ and $X_2$)

The *(stochastic) variance decomposition* of the risk $R$ is defined as

$$R = E[P(R|X_1)] + [R - E[P(R|X_1)],$$

where the risk factors $R_1$ and $R_2$ are supposed to capture the randomness caused by the sources of risk $X_1 = (X_1(t))_{0 \leq t \leq T}$ and $X_2 = (X_2(t))_{0 \leq t \leq T}$, respectively.

It follows the well-known result (not focused here):

$$Var(R) = Var(R_1) + Var(R_2)$$

**Simple example**

Let the insurer’s risk be equal to $R = X_1(T)X_2(T)$, where $X_1, X_2$ are two independent (standard) Brownian motions.

Then the variance decomposition yields two different results depending on the order of $X_1$ and $X_2$:

1) $R_1 = E[P(R|X_1)] = X_1(T)E[P(X_2(T))] = 0$

$$R_2 = R - E[P(R|X_1)] = X_1(T)X_2(T) - 0 = X_1(T)X_2(T)$$

2) $R_2 = E[P(R|X_2)] = X_2(T)E[P(X_1(T))] = 0$

$$R_1 = R - E[P(R|X_2)] = X_1(T)X_2(T) - 0 = X_1(T)X_2(T)$$

$=> *Variance decomposition is not order-invariant!*
Risk decomposition methods from literature

Overview

Properties of risk decomposition methods from existing literature

<table>
<thead>
<tr>
<th>Method</th>
<th>randomness</th>
<th>attribution</th>
<th>uniqueness</th>
<th>order invariance</th>
<th>scale invariance</th>
<th>aggregation</th>
<th>additive aggregation</th>
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<tr>
<td>Variance decomposition</td>
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<td>✓</td>
<td>X</td>
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<td>X</td>
<td>X</td>
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<tr>
<td>Taylor expansion</td>
<td>✓</td>
<td>✓</td>
<td>X</td>
<td>✓</td>
<td>X</td>
<td>X</td>
<td>X</td>
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<tr>
<td>cf. Christiansen (2007)</td>
<td></td>
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<td></td>
<td></td>
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<tr>
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<td>X</td>
<td>✓</td>
<td>✓</td>
<td>X</td>
<td>X</td>
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<tr>
<td>MRT decomposition</td>
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<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
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Modeling framework

Let all financial and demographic sources of risk be modeled by a \((k - 1)\)-dimensional Itô process:

\[
dX(t) = \theta(t) dt + \sigma(t) dW(t), \quad t \in [0, T], \quad X(0) = x_0 \in \mathbb{R}^{k-1}
\]

The number of survivors is modeled by a doubly stochastic counting process \(m - N(t)\)
- \((N(t))_{t \in [0,T]}\) number of deaths with jump intensity \((\mu(t))_{t \in [0,T]}\)

Notation
- \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) probability space
- \(T\) time horizon
- \(W\) \(d\)-dimensional Brownian motion with filtration \(\mathbb{G} = (\mathcal{G}(t))_{t \in [0,T]}\); \(\mathbb{G}\) is a sub-filtration of \(\mathbb{F}\)
- \(m \in \mathbb{N}\) denotes the initial number of policyholders
- \(\tau^i_x\) remaining life time of insured \(i = 1, \ldots, m\)
**MRT decomposition**

**Definition and properties**

**Definition**

The **MRT decomposition** of \( R = L - E^P(L) \) is defined as

\[
R = \sum_{i=1}^{k-1} \int_0^T \psi_i^W(t) dM^W_i(t) + \int_0^T \psi^N(t) dM^N(t)
\]

for some \( \mathbb{F} \)-predictable processes \( \psi_i^W(t) \) and \( \psi^N(t) \).

The processes \( M^W_i(t) \) and \( M^N(t) \) denote the martingale part of \( X_i(t) \) and \( N(t) \), respectively.

**Theorem**

Let \( L \) be \( \mathcal{F}_t \)-measurable and square integrable, \( k-1 = d \), and \( \det(\sigma(t)) \neq 0 \) for all \( t \).

Then the MRT decomposition of \( R = L - E^P(L) \) **exists** and satisfies **all properties** (i.e. randomness, attribution, uniqueness, order invariance, scale invariance, aggregation, additive aggregation).

**Further properties**

- **Applicability**: Explicit formulas for the integrands \( \psi_1^W(t), \ldots, \psi_{k-1}^W(t), \psi^N(t) \) are derived (within a life insurance context).
- **Convergence**: Unsystematic mortality risk factor \( (R_k) \) is diversifiable; all other risk factors converge to a non-zero limit as the portfolio size increases.
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MRT decomposition for a GAO

For the insurer’s (discounted) loss \( L = e^{-\int_0^T r(s)ds} \left( m - N(T) \right) gA_T \max\{a_T - \frac{1}{g}, 0\} \) from a GAO it holds (given a slightly modified stochastic model compared to slide 10):

- There exists a measurable function \( h \) such that \( h(S(T), r(T), \mu(T)) = gA_T \max\{a_T - \frac{1}{g}, 0\} \)
- The function \( f(t, S(t), r(t), \mu(t)) := E^P \left( e^{-\int_t^T (r(s) + \mu(s))ds} h(S(T), r(T), \mu(T)|\mathcal{G}_t) \right) \) is in \( \mathcal{C}^{1,2} \)

The unique MRT risk factors of the insurer’s risk \( R = L - E^P(L) = R_1 + R_2 + R_3 + R_4 \) are given by

\[
\begin{align*}
R_1 &= \int_0^T (m - N(t) -) e^{-\int_0^t r(s)ds} \frac{\partial f}{\partial x_1} (t, X(t)) \sigma_S(t) dW_S(t) \quad \text{(fund risk)} \\
R_2 &= \int_0^T (m - N(t) -) e^{-\int_0^t r(s)ds} \frac{\partial f}{\partial x_2} (t, X(t)) \sigma_r(\sqrt{t}) dW_r(t) \quad \text{(interest risk)} \\
R_3 &= \int_0^T (m - N(t) -) e^{-\int_0^t r(s)ds} \frac{\partial f}{\partial x_3} (t, X(t)) \sigma_\mu(t) \sqrt{\mu(t)} dW_\mu(t) \quad \text{(systematic mortality risk)} \\
R_4 &= \int_0^T e^{-\int_0^t r(s)ds} f(t, X(t)) dM_N(t) \quad \text{(unsystematic mortality risk)}
\end{align*}
\]
Application of MRT decomposition to annuity conversion options
Numerical results for a GMIB

Cumulative distribution functions of the total risk and the four risk factors

Relative risk contributions of the four sources of risk under different risk measures

<table>
<thead>
<tr>
<th></th>
<th>TVaR&lt;sub&gt;0.99&lt;/sub&gt;</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fund</td>
<td>89.8 %</td>
<td>96.5 %</td>
</tr>
<tr>
<td>Interest</td>
<td>8.3 %</td>
<td>3.0 %</td>
</tr>
<tr>
<td>Syst. mortality</td>
<td>1.2 %</td>
<td>0.4 %</td>
</tr>
<tr>
<td>Unsyst. mortality</td>
<td>0.7 %</td>
<td>0.1 %</td>
</tr>
</tbody>
</table>

- Fund risk dominates total risk
- Interest risk is slightly relevant in the tail
- Mortality risks are negligible
Application of MRT decomposition to annuity conversion options
Numerical results for a GAO

Relative risk contributions of the four sources of risk under TVaR_\(\alpha\) (in %) for different safety levels \(\alpha\)

- Interest risk dominates total risk
- Systematic mortality is significant
- Fund risk is particularly responsible for high risk outcomes
- Unsystematic mortality risk is negligible
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Risk analysis of annuity conversion options with a special focus on decomposing risk
Literature


